MINIMAL SURFACES IN FINITE VOLUME HYPERBOLIC 3-MANIFOLDS N AND IN $M \times \mathbb{S}^1$, M A FINITE AREA HYPERBOLIC SURFACE.

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Abstract. We consider properly immersed finite topology minimal surfaces Σ in complete finite volume hyperbolic 3-manifolds N, and in $M \times \mathbb{S}^1$, where M is a complete hyperbolic surface of finite area. We prove Σ has finite total curvature equal to 2π times the Euler characteristic $\chi(\Sigma)$ of Σ, and we describe the geometry of the ends of Σ. .

1. Introduction

Let N denote a complete hyperbolic 3-manifold of finite volume. An end $\mathcal{M}$ of N is modeled on the quotient of a horoball of the hyperbolic 3-space $\mathbb{H}^3$, by a $\mathbb{Z}^2$ parabolic subgroup of the isometry group of $\mathbb{H}^3$ leaving the horoball invariant. More precisely we consider the model of the half-space of $\mathbb{H}^3 = \{(x, y, t) \in \mathbb{R}^3; y > 0\}$ with the metric $ds^2 = \frac{dx^2+dy^2+dt^2}{y^2}$. Then an end of N has a sub-end isometric to

$$\mathcal{M}(-1) = \{(x, y, t) \in \mathbb{R}^3; y \geq y_0 > 0\}$$

modulo a $\mathbb{Z}^2$-parabolic subgroup of isometries of $\mathbb{H}^3$ leaving the planes $\{y = c\}$ invariant. The horosphere y=constant quotient to tori $\mathbb{T}(y)$ in $\mathcal{M}(-1)$; $\mathbb{T}(y)$ has constant mean curvature one. Let c be a compact geodesic of $\mathbb{T}(1)$. Then $A(-1) = \{(c, t); t \geq 1\}$ is a minimal annulus immersed in $\mathcal{M}(-1)$, which we will call a standard cusp-end in $\mathcal{M}(-1)$

A complete surface M of constant curvature $K = -1$ and finite area has finite total curvature hence M is conformally diffeomorphic to a compact surface punctured in a finite number of points. Each end of M (called a cusp end), denoted $\mathcal{C}$, is an annular end isometric to the quotient of a horodisk H in the hyperbolic plane $\mathbb{H}^2$ by a parabolic isometry ψ.

To describe the geometry of such ends we model $\mathbb{H}^2$ by the upper half plane

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2; y > 0\}$$

with the metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. Then a cusp end $\mathcal{C}$ of M is isometric to $H/[\psi]$, where $H = \{(x, y) \in \mathbb{R}^2; y \geq 1\}$ is a horodisk and $\psi(x, y) = (x + \tau, y)$, for some $\tau \neq 0$.

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In $M \times \mathbb{S}^1$, with the product metric, the ends become $\mathcal{M} := \mathcal{C} \times \mathbb{S}^1$, and are foliated by constant mean curvature tori $\mathbb{T}(y_1) = c(y_1) \times \mathbb{S}^1$, where $c(y_1) = \{(x, y) \in \mathbb{H}^2; y = y_1\}/[\psi]$. We consider $\mathbb{S}^1 = \mathbb{R}/T(h)$, $T(h)$ the translation of $\mathbb{R}$ by some $h > 0$ and

$$\mathcal{M} = \cup_{y \geq y_0} \mathbb{T}(y) = (H/[\psi]) \times (\mathbb{R}/T(h)) = \{(x, y, t) \in \mathbb{R}^3; y \geq y_0 \geq 1\}/[\psi, T(h)].$$

Thus the ends of N and those of $M \times \mathbb{S}^1$ share many properties. Both are parametrized by the same half-space of $\mathbb{R}^3$, and foliated by constant mean curvature tori $\mathbb{T}(y)$ (curvature one half in $\mathcal{M}$ and one in $\mathcal{M}(-1)$). $\mathcal{M}(-1)$ has constant sectional curvature -1 and the tori $\mathbb{T}(y)$ shrink exponentially when one flows by the geodesics y increasing. In $\mathcal{M}$, the horizontal cycles $c(y)$ shrink exponentially along the y increasing flow and the t cycles are of constant length h. Subsequently we will develop the geometry of surfaces in these ends.

Now let Σ be a properly embedded minimal surface in N or $M \times \mathbb{S}^1$ of finite topology; so that Σ has a finite number of annular ends $\{A_j\}$ for $1 \leq j \leq k$. Since Σ is proper, each end A_j of Σ is in some end $\mathcal{M}$ of $M \times \mathbb{S}^1$ or in some end $\mathcal{M}(-1)$ of N. We denote by E a connected component of a lift of an end A of Σ, E in $\mathbb{H} \times \mathbb{R}$ or $\mathbb{H}^3$.

We will now describe the model ends of minimal annuli in $\mathcal{M}$ and $\mathcal{M}(-1)$. In $\mathcal{M}(-1)$ the model end is the standard cusp end $A(-1)$ we previously defined.

In $\mathcal{M}$, there are essentially three model ends. In $\mathcal{M}$, we define $A_{(p,q)}$ to be the annular end that is the quotient of a (euclidean) half-plane $E_{(p,q)}$ orthogonal to the plane $\{(x, y, t) \in \mathbb{R}^3; y = 1\}$ and of slope $qh/p\tau$. For $(p, q) = (1, 0)$, the end

$$E_{(1,0)}(t_0) = \{(x, y, t) \in \mathbb{R}^3; y \geq 1, t = t_0\} \text{ and } A_{(1,0)} = E_{(p,q)}/[\psi]$$

is a cusp end of M(horizontal). For $(p, q) = (0, 1)$, it is the product of a horizontal geodesic ray of M and $\mathbb{S}^1$. The end

$$E_{(0,1)}(x_0) = \{(x, y, t) \in \mathbb{R}^3; y \geq 1, x = x_0\} \text{ and } A_{(0,1)} = E_{(0,1)}/T(h).$$

For $(p, q) \neq \{(0, 1), (1, 0)\}$, we think of $A_{(p,q)}$ as a helicoid with axis at the cusp at infinity. It is the quotient of

$$E_{(p,q)}(c_0) = \{(x, y, t) \in \mathbb{R}^3; y \geq 1, p\tau t - qhx = c_0\} \text{ and } A_{(p,q)} = E_{(p,q)}/[\psi, T(h)].$$

We will prove that a properly immersed annular end A in $\mathcal{M}$ or in $\mathcal{M}(-1)$ has finite total curvature and is asymptotic to a standard end $A_{(p,q)}$ in $\mathcal{M}$ or a standard cusp end $A(-1)$ in $\mathcal{M}(-1)$. The main theorem of the paper is:

Theorem 1.1. *Consider a complete surface M with curvature $K = -1$ and finite area and N a complete hyperbolic 3-manifold of finite volume. Let Σ be a properly immersed minimal surfaces in N or in $M \times \mathbb{S}^1$ with finite topology. Then the surface Σ has finite total curvature and each end A of Σ is asymptotic to a standard cusp-end $A(-1)$ in $\mathcal{M}(-1)$ or to a standard end $A_{(p,q)}$ in $\mathcal{M}$:*

(i) $A_{(p,0)}$ *a horizontal cusp* $\mathcal{C} \times \{t_0\}$
(ii) $A_{(0,q)}$ *a vertical plane* $\gamma \times \mathbb{S}^1$
(iii) $A_{(p,q)}$ *a helicoidal end with axis at infinity.*

Moreover

$$\int_\Sigma K dA = 2\pi\chi(\Sigma).$$

Corollaries of the theorem 1.1 Combining the formula for the total curvature of Σ in theorem 1.1, with the Gauss equation, we obtain topological obstructions for the existence of proper minimal immersions of a finite topology surface Σ into N or $M \times \mathbb{S}^1$, of a given topology.

For example, there is no such proper minimal immersion of a plane $\mathbb{R}^2$ into N or $M \times \mathbb{S}^1$. In N there is no proper minimal immersion of the sphere $\mathbb{S}^2$ with n punctures; $n = 0, 1$, or 2. A proper minimal immersion of $\mathbb{S}^2$ with two punctures (an annulus) in $M \times \mathbb{S}^1$, is necessarily $\gamma \times \mathbb{S}^1$, γ a complete geodesic of M.

More generally, suppose Σ is an orientable surface of genus g with n punctures, $n \geq 0$. Then $\chi(\Sigma) = 2-2g-n$, so if Σ can be properly minimally immersed in N or $M \times \mathbb{S}^1$, it follows from theorem 1.1 and the Gauss equation

$$\int_\Sigma K_\Sigma = 2\pi(2-2g-n) = \int_\Sigma K_e + \int_\Sigma K_\sigma,$$

where K_e and K_σ are the extrinsic and sectional curvatures of Σ respectively. Since $-1 \leq K_\sigma \leq 0$ in $M \times \mathbb{S}^1$, $K_\sigma = -1$ in N, and $K_e \leq 0$, we have

$$2 - 2g - n \leq 0$$

and equality if and only if $K_e = K_\sigma = 0$.

This equality cannot occur in N (since $K_\sigma = -1$) and equality in $M \times \mathbb{S}^1$ yields Σ is vertical and g is 0 or 1. When $g = 0$, then $n = 2$ and $\Sigma = \gamma \times \mathbb{S}^1$ γ a complete, non compact geodesic of M.

When $2 < 2g + n$ then if $g = 0$, one can not have $n \leq 2$. So excluding the equality case we discussed above, there is no proper minimal immersion of $\mathbb{S}^2$ with $0, 1$, or 2 punctures, in N or $M \times \mathbb{S}^1$.

In N, one obtains an area estimate. If Σ is properly minimally immersed in N then

$$2\pi(2-2g-n) = \int_\Sigma K_e - |\Sigma|,$$

so $|\Sigma| = \int_\Sigma K_e + 2\pi(2g+n-2) \leq 2\pi(2g+n-2)$ and equality precisely when Σ is totally geodesic. Do such totally geodesic immersions exists in N?

We have $0 < 2g + n - 2$, so if $2g + n - 2 \leq 0$, the immersion Σ does not exist in N. If $2g > 2 - n$, can Σ be properly minimally immersed in N?.

The paper is organized as follows. We will begin considering surfaces in $M \times \mathbb{S}^1$. First we describe some examples of properly embedded minimal surfaces of finite topology in $M \times \mathbb{S}^1$. We start with M a 3-punctured sphere, then M a sphere with $2n$ punctures, and M a once punctured torus. We hope to convey to the reader the wealth of interesting examples in these spaces.

In section 2, we describe some properties of the standard examples $A_{(p,q)}$ in the cusp ends $\mathcal{M}$ of $M \times \mathbb{S}^1$. We construct auxiliary minimal surfaces needed for the sequel.

In section 3 we begin the study of a lift $E \subset H \times \mathbb{R}$ of an annular end A of $\Sigma \cap \mathcal{M}$. We prove that a subend of A is trapped between two standard ends $A_{(p,q)}$ that are close at infinity; "close" will be defined later.

In section 4, we study compact annuli that we will use in the proof of the theorem. In section 5, we study the limit of a family of Scherk type graphs in $\mathbb{H}^2$ which are converging to 0 and we use this sequence to prove that the third coordinate of an end of type $(1,0)$ has a limit at infinity.

In section 6, we prove that a trapped subend of A is a killing graph, hence has bounded curvature. Then in sections 7,8, and 9, we prove the main theorem.

2. Examples in $M \times \mathbb{S}^1$

The first examples in $M \times \mathbb{S}^1$ that come to mind are the horizontal slices $\Sigma = M \times \{c\}$ and the vertical annuli (or totally geodesic tori), $\Sigma = \gamma \times \mathbb{S}^1$, γ a complete geodesic (perhaps compact) of M.

We describe five examples; M will be a sphere with three or four punctures or a once punctured torus, and have a complete hyperbolic metric of finite area. Denote by $\mathbb{S}\mathrm{ph}(k), k = 3$ or 4 such a hyperbolic sphere and $\mathbb{T}\mathrm{or}(1)$ a once punctured hyperbolic torus.

Example 1. Σ an embedded minimal surface in $\mathbb{S}\mathrm{ph}(3) \times \mathbb{S}^1$ with three ends; two helicoidal and the other horizontal. The domains and notation we now introduce will be used in all the examples we describe.

Let Γ be the ideal triangle in the disk model of $\mathbb{H}^2$ with vertices $A = (0,1), B = (0,-1), C = (-1,0)$ and sides a, b, c as indicated in figure 1.

Let Σ, be the minimal graph over the domain D bounded by Γ, taking the values 0 on b and c and $h > 0$ on a. Extend Σ, to an entire minimal graph $\tilde{\Sigma}$ over $\mathbb{H}^2$ by rotation by π in all the sides of Γ, and the sides of the triangles thus obtained.

In figure 2, we indicate some of the reflected triangles and the values of the graph $\tilde{\Sigma}$ on their sides.

Let D be the domain bounded by Γ. Let ψ_A be the parabolic isometry with fixed point A which takes the geodesic c to c_1 and a to a_1; $\psi_A = R_{c_1} R_a$, where R_γ denotes reflection in the geodesic γ. Let ψ_B be the parabolic isometry of $\mathbb{H}^2$ leaving B fixed, taking b to b_1 and a to c_2; $\psi_B = R_{b_1} R_a$.

Notice that the group of isometries of $\mathbb{H}^2 \times \mathbb{R}$, generated by $T(2h) \circ \psi_A$ and $T(2h) \circ \psi_B$, leaves $\tilde{\Sigma}$ invariant.

Let M be the 3-punctured sphere obtained by identifying the sides of $D \cup R_a(D)$ by ψ_A, ψ_B (c with c_1, b with b_1). M is hyperbolic and has finite area.

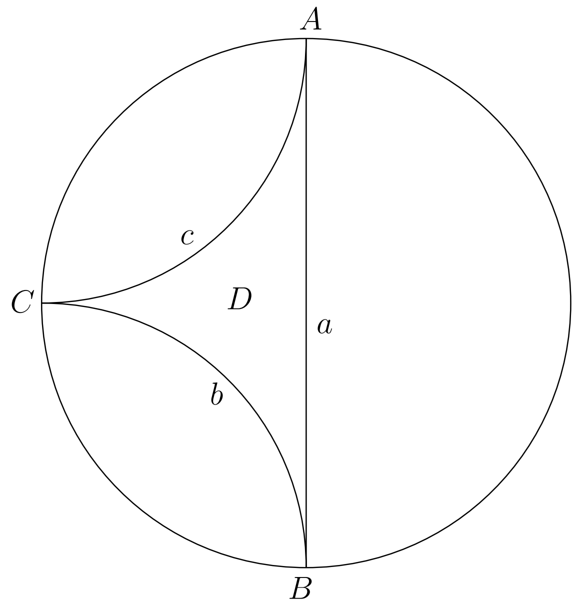

FIGURE 1. Ideal triangle (ABC) in $\mathbb{H}^2$

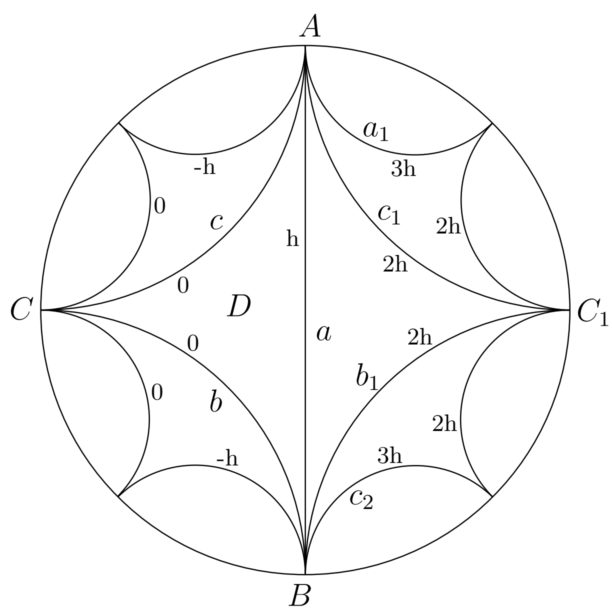

FIGURE 2. Value of the graph $\tilde{\Sigma}$ on geodesics

Let Σ_2 be the graph in $\tilde{\Sigma}$ over $D \cup R_a(D)$. Then the multi-graph $\cup_{k\in\mathbb{Z}}\mathbb{T}_{k2h}(\Sigma_2)$ passes to the quotient $M \times (\mathbb{R}/T(2h))$ to give a complete embedded minimal surface Σ with 3-ends; two helicoidal and the other horizontal. Σ is a 3-punctured sphere, has total curvature -2π and Σ is stable (Σ is transverse to the killing field $\partial/\partial t$).

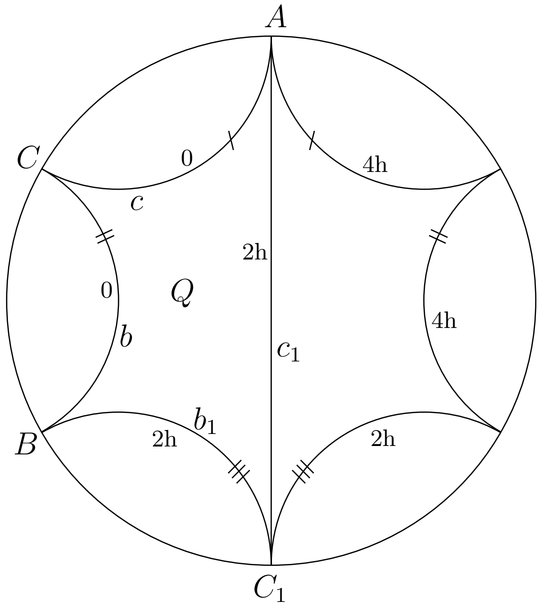

FIGURE 3. M is a 4-punctured sphere

Example 2. Σ an embedded minimal surface in $\mathbb{S}\mathrm{ph}(4) \times \mathbb{S}^1$, the sphere with four ends. Let Q be the ideal quadrilateral $D \cup R_a(D)$, and define $F = Q \cup R_{c_1}(Q)$. Let M be the quotient of F obtained by identifying the sides of ∂F as follows:

(1) Identify c with $R_{c_1}(c)$ by the parabolic isometry at A taking c to $R_{c_1}(c)$,
(2) Identify b with $R_{c_1}(b)$ by the hyperbolic isometry taking b to $R_{c_1}(b)$ and,
(3) Identify b_1 with $R_{c_1}(b_1)$ by the parabolic isometry at C_1 taking b_1 to $R_{c_1}(b_1)$.

M is a 4-punctured sphere. A more (apparently) symmetric picture of M is obtained by changing the picture by the isometry taking A to A and C_1 to B as indicated in the figure 3.

Then the graph of $\tilde{\Sigma}$ over F yields a embedded minimal surface Σ in $(M \times \mathbb{R})/[T(4h)]$. Σ has two horizontal ends and two helicoidal ends of type $E(1,1)$. Σ is also stable.

Example 3. A compact singly periodic Scherk surface; M a once punctured torus. This surface is constructed in [8]; we describe it here. Let $Q = D \cup R_a(D)$ and γ_1, γ_2 be minimizing geodesics joining opposite sides of D; figure 4. In $\mathbb{H}^2 \times \mathbb{R}$, we desingularize the intersection of the planes $\gamma_1 \times \mathbb{R}$ and $\gamma_2 \times \mathbb{R}$ in the usual manner to create a Scherk surface invariant under a vertical translation. We describe this. Let α and β be the segments of γ_1, γ_2 in the first and fourth quadrants respectively. Form a polygon in $\mathbb{H}^2 \times \mathbb{R}$ by joining to $(\alpha \times \{h\}) \cup (\beta \times \{0\})$ by the two vertical segments joining $\alpha(\Gamma) \times \{0\}$ to $\alpha(\Gamma) \times \{h\}$, and joining $\beta(\Gamma) \times \{0\}$ to $\beta(\Gamma) \times \{h\}$; $\alpha(\Gamma)$ denotes the endpoint of α on Γ (similarly for $\beta(\Gamma)$). This polygon bounds a least area minimal disk D_1; figure 5

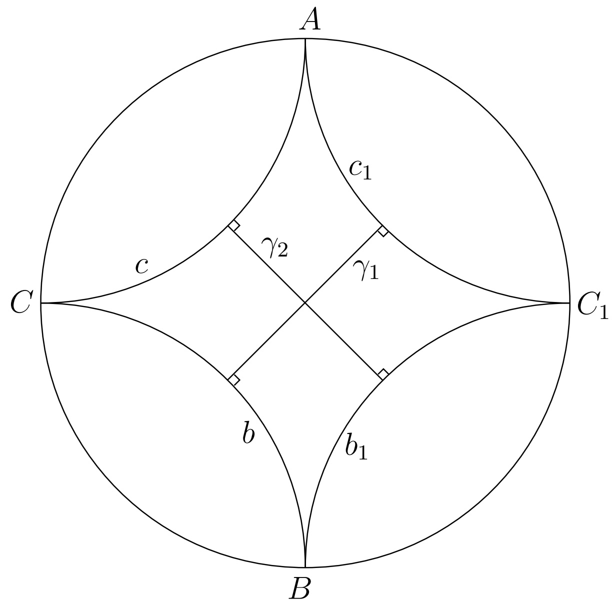

FIGURE 4. M is a once punctured torus

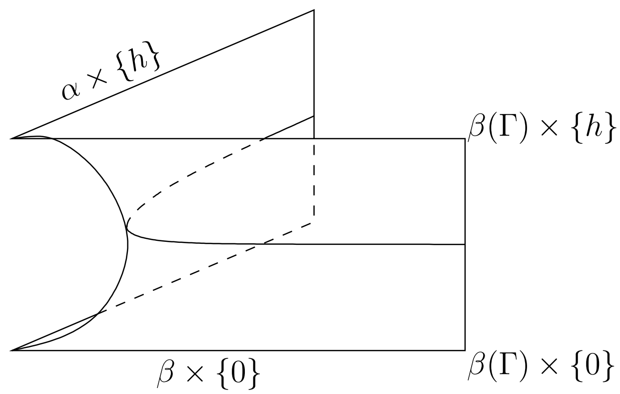

FIGURE 5. A least area disk D_1

Successive symmetries in all the horizonntal sides yields a "Scherk" type surface in $\mathbb{H}^2 \times \mathbb{R}$ bounded by 4 vertical geodesics, invariant by vertical translation by $2h$.

We now identify opposite sides of Q by the hyperbolic translations $T(\gamma_1), T(\gamma_2)$, along γ_1 and γ_2. This gives a once punctured torus M. The Scherk surface passes to the quotient to give a compact minimal surface Σ with $\partial\Sigma = \emptyset$, in $M \times (\mathbb{R}/T(2h))$.

Example 4. A singly periodic Scherk surface with 4 vertical annular ends; M a once punctured torus. Now we "rotate" example 4 by $\pi/4$. Let α_1, α_2 be the complete geodesics joining the opposite vertices of ∂Q; α_1, α_2 are the x and the y axis in the

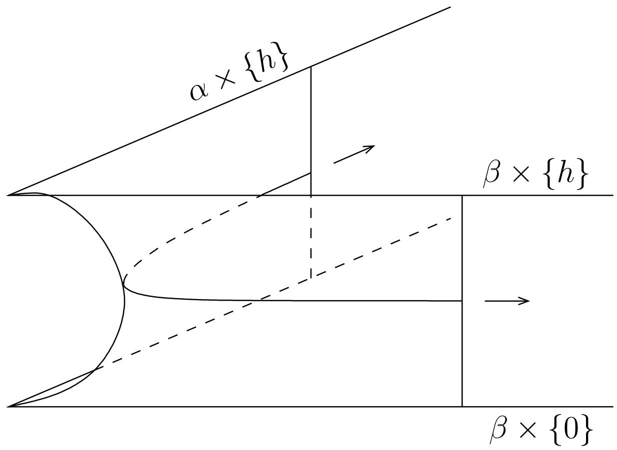

FIGURE 6. A piece of singly periodic Scherk surface with 4 vertical annular ends

unit disc model. Again we construct Plateau disks bounded by the polygon of figure 6.

We know that when the vertical geodesic segments diverge along β, the plateau solutions converge to a complete embedded surface in $\mathbb{H}^2 \times \mathbb{R}$ with boundary $\{(x,0)/x \geq 0\} \cup \{(y,0)/y \geq 0\} \cup \{(x,h)/x \geq 0\} \cup \{(y,h)/y \geq 0\}$.

The symmetries of this surface along all the edges yields a singly periodic Scherk surface in $Q \times \mathbb{R}$, invariant by $T(2h)$.

As in example 4, we identify the opposite sides of Q by hyperbolic translations to obtain a torus M with one puncture. This gives the Scherk surface Σ in $M \times \mathbb{R}/[T(2h)]$, with four vertical annular ends.

We remark that one can quotient ∂Q by parabolic isometries to obtain this Scherk surface in $M \times \mathbb{S}^1$ where M is now a 4-punctured sphere.

Example 5. A helicoid with helicoidal ends in $M \times \mathbb{S}^1$, M a once punctured torus. It is convenient to describe this example in $M \times \mathbb{S}^1$ where M is the once punctured torus obtained from the ideal quadrilateral Q_1 in $\mathbb{H}^2$ with the 4 vertices $(\pm\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}})$, by identifying opposite sides.

Let S be the third quadrant of Q_1: $S = \{(x,y) \in Q_1; x \leq 0, y \leq 0\}$. For $h > 0$, let Σ_1 be the minimal graph over S with boundary values indicated in figure 7.

Let Σ_3 be the reflection of Σ_1 through β (cf figure 7); Σ_3 is between heights h and $2h$ and is a graph over the second quadrant of Q_1. Then rotate $\Sigma_1 \cup \Sigma_3$ by π through the vertical axis between $(0,0)$ and $(0,2h)$, to obtain $\Sigma_2 \cup \Sigma_4$; Σ_4 is a graph over the fourth quadrant of Q_1. Σ is the union of the four pieces Σ_1, through Σ_4, identified along the boundaries as follows.

First we consider identifying opposite sides of Q_1 be the hyperbolic translations sending the opposite side to the other. Then we can quotient by $T(2h)$ or by $T(4h)$. The

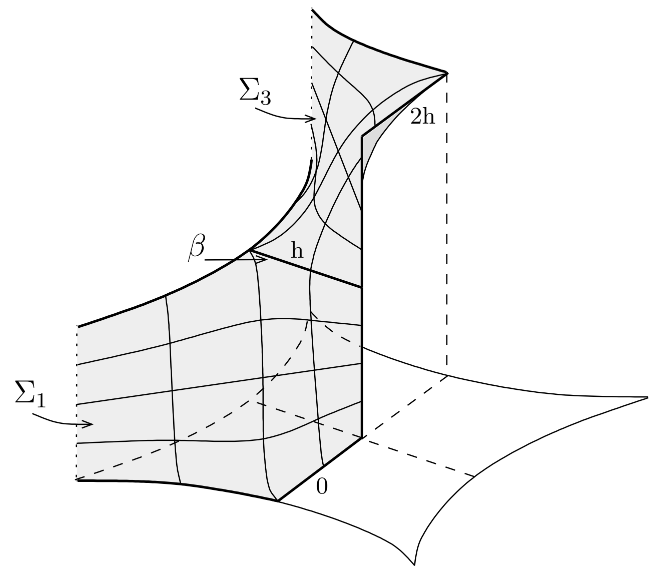

FIGURE 7. Σ_1 be a minimal graph over S

first quotient gives an non orientable surface in $M \times \mathbb{S}^1$ with one helicoid type end. The second gives an orientable surface of total curvature -8π with two helicoidal type ends (it is a double cover of the first example). Topologically the first example is the connected sum of a once punctured torus and a projective plane. The second surface is 2 punctured orientable surface of genus two.

The reader can see the helicoidal structure of Σ by going along a horizontal geodesic on Σ at $h = 0$, from one puncture to the other. Then spiral up Σ along a helice going to the horizontal geodesic at height h. Continue along this geodesic to the other (it's the same) puncture and spiral up the helices on Σ to height $2h$. If we do this right, we are back where we started.

3. BARRIERS IN $M \times \mathbb{S}^1$

We construct barriers by solving the mean curvature equation of ruled surfaces. These barriers will be used to prove the Trapping Theorem in section 4. In the model $\mathbb{H}^2 \times \mathbb{R} = \{(x, y, t) \in \mathbb{R}^3; y > 0\}$, we consider surfaces

$$X : (u, v) \to (u, \alpha(v), v + \lambda u)$$

for a $\mathcal{C}^2$ real positive function of one variable $v \to \alpha(v)$ defined on some interval I.

Lemma 3.1. *The mean curvature H of the surface $X : (u, v) \to (u, \alpha(v), v + \lambda u)$ immersed in $\mathbb{H}^2 \times \mathbb{R} = \{(x, y, t) \in \mathbb{R}^3; y > 0\}$ with the metric $ds^2 = \frac{dx^2+dy^2}{y^2} + dt^2$ is given by*

$$2H = \frac{-\alpha^2}{Z^3}\left(\alpha''(1+\lambda^2\alpha^2) + \alpha(1+\lambda^2(\alpha')^2)\right)$$

Proof. In the model of $\mathbb{H}^2 \times \mathbb{R} = \{(x,y,t) \in \mathbb{R}^3; y > 0\}$ with the metric $ds^2 = \frac{dx^2+dy^2}{y^2} + dt^2$, the non zero terms of the connection are given by

$$\nabla_{\partial/\partial x}\partial/\partial x = \tfrac{1}{y}\partial/\partial y \quad \nabla_{\partial/\partial y}\partial/\partial y = -\tfrac{1}{y}\partial/\partial y$$

$$\nabla_{\partial/\partial x}\partial/\partial y = \nabla_{\partial/\partial y}\partial/\partial x = -\tfrac{1}{y}\partial/\partial x$$

The tangent space is generated by

$$dX(\partial/\partial u) = E_1 = \partial/\partial x + \lambda\partial/\partial t = (1,0,\lambda)$$
$$dX(\partial/\partial v) = E_2 = \alpha'(v)\partial/\partial y + \partial/\partial t = (0,\alpha'(v),1)$$

The direct unit normal vector is given by $N = V/Z$ with $V = E_1 \wedge E_2$,

$$V = -\lambda\alpha'(v)\alpha(v)^2\partial/\partial x - \alpha(v)^2\partial/\partial y + \alpha'(v)\partial/\partial t = (-\lambda\alpha'(v)\alpha(v)^2, -\alpha(v)^2, \alpha'(v))$$
$$Z^2 = |V|^2 = (\alpha'(v))^2(1+\lambda^2\alpha(v)^2) + \alpha(v)^2.$$

We compute the mean curvature by the divergence formula

$$-2H = \operatorname{div}(N) = \operatorname{div}\left(\frac{V}{Z}\right) = \frac{1}{Z^3}(Z^2\operatorname{div}(V) - \frac{1}{2}V.(Z^2)).$$

We compute the first term

$$\begin{aligned}\operatorname{div}(V) = &-\frac{\partial}{\partial x}(\lambda\alpha^2(v)\alpha'(v)) - \lambda\alpha^2(v)\alpha'(v)\operatorname{div}\left(\frac{\partial}{\partial x}\right)\\ &-\frac{\partial}{\partial y}(\alpha^2(v)) - \alpha^2(v)\operatorname{div}\left(\frac{\partial}{\partial y}\right) + \frac{\partial}{\partial t}(\alpha'(v)) + \alpha'(v)\operatorname{div}\left(\frac{\partial}{\partial t}\right).\end{aligned}$$

Using $\operatorname{div}\left(\frac{\partial}{\partial x}\right) = \operatorname{div}\left(\frac{\partial}{\partial t}\right) = 0$ and $\operatorname{div}\left(\frac{\partial}{\partial y}\right) = -\frac{2}{y}$ with $\alpha(v) = y$ and $v = t - \lambda x$, a direct computation gives

$$\begin{aligned}\operatorname{div}(V) &= \lambda^2\alpha^2\alpha'' - 2\alpha - \alpha^2\left(-\frac{2}{\alpha}\right) + \alpha''\\ &= (1+\lambda^2\alpha^2)\alpha''\end{aligned}$$

For the second term

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial x}(Z^2) &= -\lambda\alpha'\alpha''(1+\lambda^2\alpha^2)\\ \frac{1}{2}\frac{\partial}{\partial y}(Z^2) &= \alpha(1+\lambda^2\alpha'^2)\\ \frac{1}{2}\frac{\partial}{\partial t}(Z^2) &= \alpha'\alpha''(1+\lambda^2\alpha^2)\end{aligned}$$

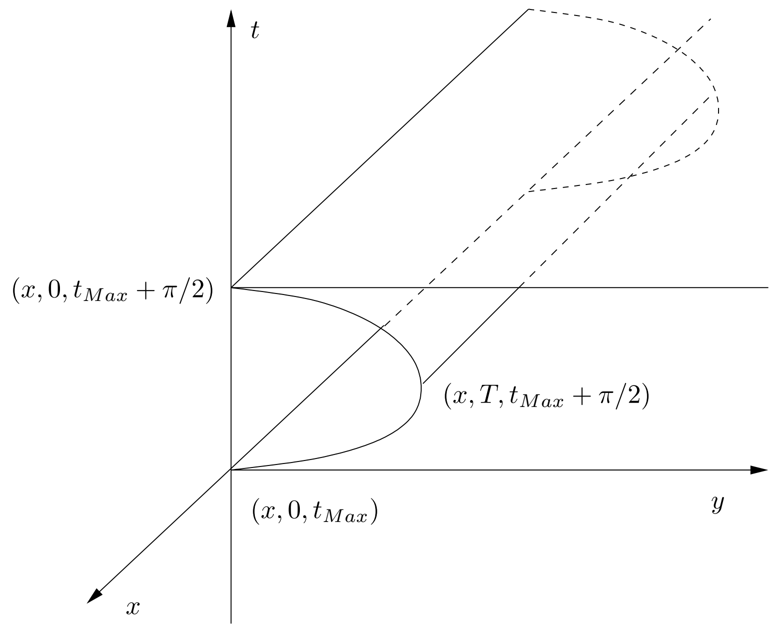

FIGURE 8. S^0, Surfaces foliated by horizontal horocycles

Hence $\frac{1}{2}V.(Z^2) = (1 + \lambda^2\alpha^2)\lambda^2\alpha^2\alpha'^2\alpha'' - \alpha^3(1 + \lambda^2\alpha'^2) + \alpha'^2\alpha''(1 + \lambda^2\alpha^2)$. Finally we obtain

$$\begin{aligned}
\operatorname{div}(N) &= \operatorname{div}(V/Z) \\
&= \frac{1}{Z^3}[(1 + \lambda^2\alpha^2)\alpha''(\alpha^2 + \alpha'^2 + \lambda^2\alpha^2\alpha'^2) \\
&\quad - (1 + \lambda^2\alpha^2)\lambda^2\alpha^2\alpha'^2\alpha'' + \alpha^3(1 + \lambda^2\alpha'^2) + \alpha'^2\alpha''(1 + \lambda^2\alpha^2)] \\
&= \frac{\alpha^2}{Z^3}[\alpha''(1 + \lambda^2\alpha^2) + \alpha(1 + \lambda^2\alpha'^2)] = -2H
\end{aligned}$$

□

We study the geometry of surfaces $X : (u, v) \to (u, \alpha(v), v + \lambda u)$ which are minimal. We notice they are ruled surfaces foliated by curves $v \to (0, \alpha(v), v)$ where $\alpha \in \mathcal{C}^2(I)$, $\alpha > 0$, $\lambda \geq 0$.

The first case solves the equation when $\lambda = 0$. The solution $\alpha(v) = T \sin v$ gives the family of minimal surfaces up to vertical translation

$$S_T^0 = \{(u, T \sin v, v) \in \mathbb{R}^3; u \in \mathbb{R}, v \in [0, \pi]\}$$

This surface is foliated by horizontal horocycles $u \to (u, \alpha(v), v)$ and is described in Hauswirth [5], then by Toubiana and Sa Earp [7], Daniel [3] and Mazet, Rodriguez, Rosenberg [8](see figure 8). By the nature of the curve $v \to (0, T \sin v, v)$, the surfaces S_T^0 foliate the slab $S = \{(x, y, t) \in \mathbb{R}^3; 0 < y, 0 \leq t \leq \pi\}$.

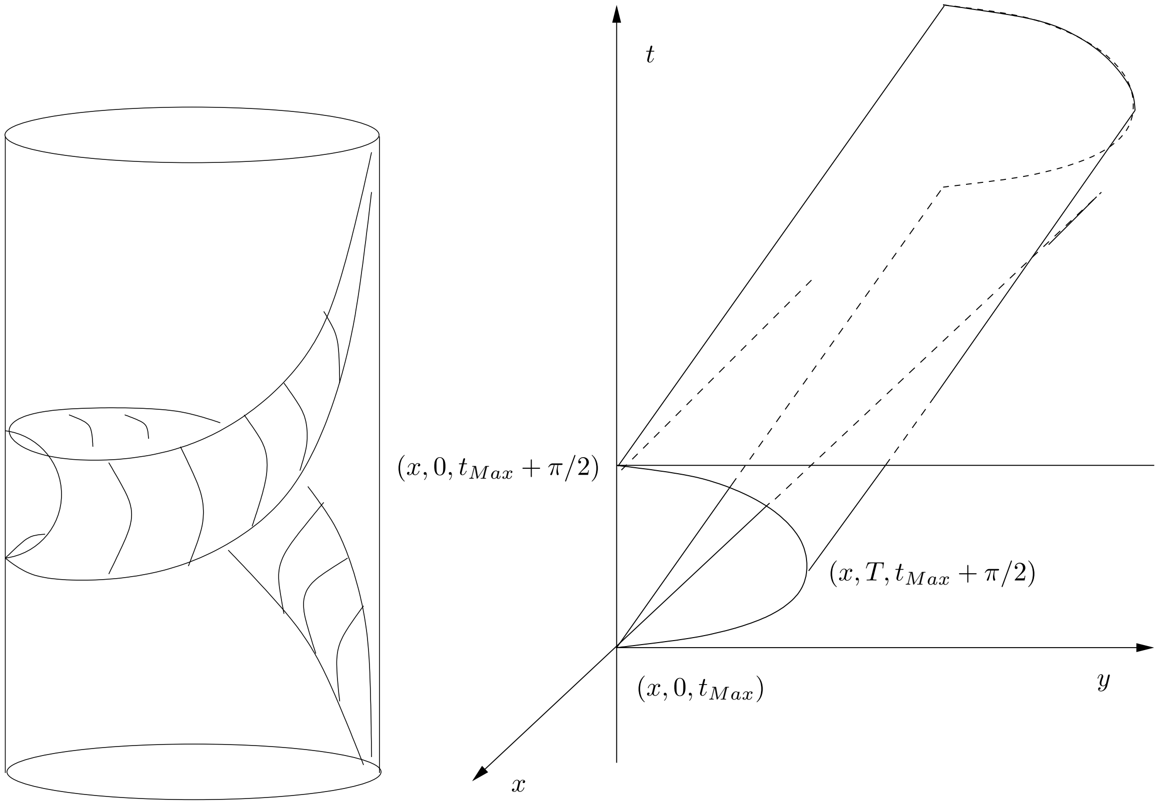

FIGURE 9. Surfaces S_T^λ in the disk model and half-space model of $\mathbb{H} \times \mathbb{R}$

The general case $\lambda \neq 0$ depends on a function α , a solution of the equation

$$\alpha''(1 + \lambda^2\alpha^2) + \alpha(1 + \lambda^2(\alpha')^2) = 0.$$

This equation has a first integral $(1 + \lambda^2\alpha'^2)(1 + \lambda^2\alpha^2) = T$ for some fixed constant $T > 1$. Since $\alpha'' < 0$ the curves $v \to (0, \alpha(v), v)$ are convex. For fixed $T > 1$, the function $\alpha_T(v)$ has its maximum value at $\sup \alpha_T(v) = \lambda^{-1}\sqrt{T-1}$. The function α_T is positive on a set $[0, v_0(T)]$. The solution $\alpha_T(v)$ with initial data $\alpha_T(0) = 0$ and $\alpha'(0) = \lambda^{-1}\sqrt{T-1}$ defines a one-parameter family of minimal surfaces

$$S_T^\lambda = \{(u, \alpha_T(v), v + \lambda u) \in \mathbb{R}^3; u \in \mathbb{R}, v \in [0, v_0(T)]\}.$$

For large values of T and fixed constant $M > 0$, we look for the set of values where $0 \leq \alpha_T(v) \leq M$. On this set we remark that

$$\alpha'^2 = \lambda^{-2}\left(\frac{T}{1 + \lambda^2\alpha^2} - 1\right) \geq \lambda^{-2}\left(\frac{T}{1 + \lambda^2 M^2} - 1\right)$$

which implies that $\alpha'_T(v) \to \infty$ when $T \to \infty$. The part of the curve $(0, \alpha_T(v), v)$ contained in $0 < y \leq M$ converges to the half geodesic $\{(0, y, 0) \in \mathbb{R}^3; 0 < y \leq M\}$.

We summarize this discussion in the figure 9 and we will use the following lemma:

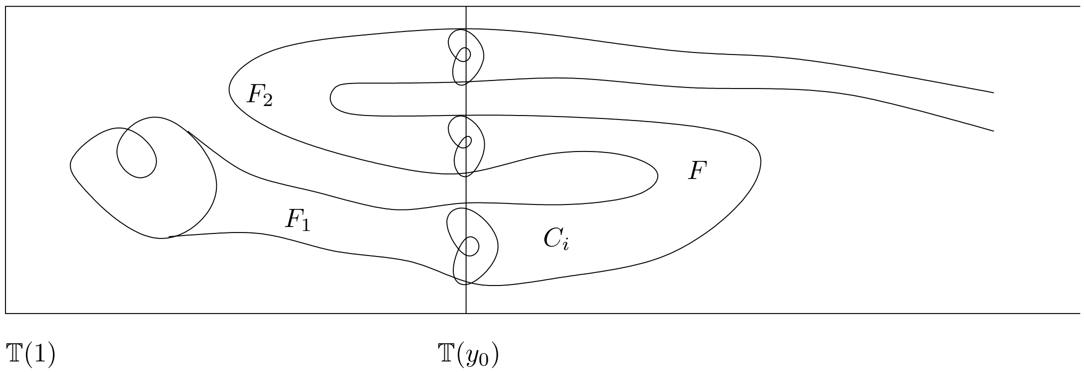

FIGURE 10. An annular end in $\mathcal{M}$

Lemma 3.2. *a) The family of surfaces S_T^0, foliates the slab $\mathbb{H}^2 \times [0, \pi]$ and when T goes to infinity the surfaces S_T^0 converge on compact sets to the horizontal section $\mathbb{H}^2 \times \{0\}$.*

b) The one-parameter family of surfaces S_T^λ converges on compact sets to $\{(x, y, t) \in \mathbb{R}^3; y > 0$ and $t = \lambda x\}$.

4. TRAPPING THEOREM FOR MINIMAL ENDS

We consider a minimal surface Σ of finite topology, hence each end of Σ is an annular end. Since Σ is properly immersed, each end A_0 of Σ is contained in some end $\mathcal{M}$ of $M \times \mathbb{S}^1$.

Lemma 4.1. *There is $y_0 \geq 1$ and a sub-end A of A_0 such that $\partial A \subset \mathbb{T}(y_0)$, A is transverse to $\mathbb{T}(y_0)$ and $A \subset \cup_{y \geq y_0} \mathbb{T}(y)$.*

Proof. Since Σ is properly immersed each end of Σ has a subend A_0 contained in some $\mathcal{M} = \cup_{y \geq 1} \mathbb{T}(y)$. A_0 is transverse to almost every $\mathbb{T}(y)$ so let $y_0 > 1$ be such that $\partial A_0 \subset \cup_{1 \leq y < y_0} \mathbb{T}(y)$, and A_0 is transverse to $\mathbb{T}(y_0)$. Then $A_0 \cap \mathbb{T}(y_0) = C_1 \cup ... \cup C_k$, each C_j an immersed Jordan curve in $\mathbb{T}(y_0)$.

A_0 is proper so $A_0 \cap \mathbb{T}(y) \neq \emptyset$ for large y. Hence at least one of the C_j is not null homotopic in A_0. Observe that there is at most one such C_j. For if C_i and C_j are not trivial then they bound a compact domain F in A_0 disjoint from ∂A_0. F cannot be contained in $\cup_{1 \leq y \leq y_0} \mathbb{T}(y)$ since then F would touch some $\mathbb{T}(y_1), y_1 < y_0$, on the mean convex side of $\mathbb{T}(y_1)$, a contradiction. So $F \subset \cup_{y \geq y_0} \mathbb{T}(y)$. But then, ∂A_0 and C_i or C_j (C_i say) would bound a compact F_1 on A_0, $F_1 \cap C_j = \emptyset$. $A_0 - F_1$ is an annular sub-end of A_0 with boundary C_i contained in $\partial \mathbb{T}(y_0)$. Since $A_0 - F_1$ intersects $\mathbb{T}(y_0)$ also at C_j, there is a compact domain F_2 of $A_0 - F_1$ contained in $\cup_{1 \leq y \leq y_0} \mathbb{T}(y)$ with $\partial F_2 \subset \mathbb{T}(y_0)$; a contradiction; see figure 10.

Now it is clear that if each C_ℓ, $\ell \neq i$ bounds a disk D on A_0 that is contained in $\cup_{y \geq y_0} \mathbb{T}(y)$. It follows that the connected component of A in $\cup_{y \geq y_0} \mathbb{T}(y)$ that has C_i in its boundary has no other C_ℓ, $l \neq i$, in its boundary. This proves the lemma. $\square$

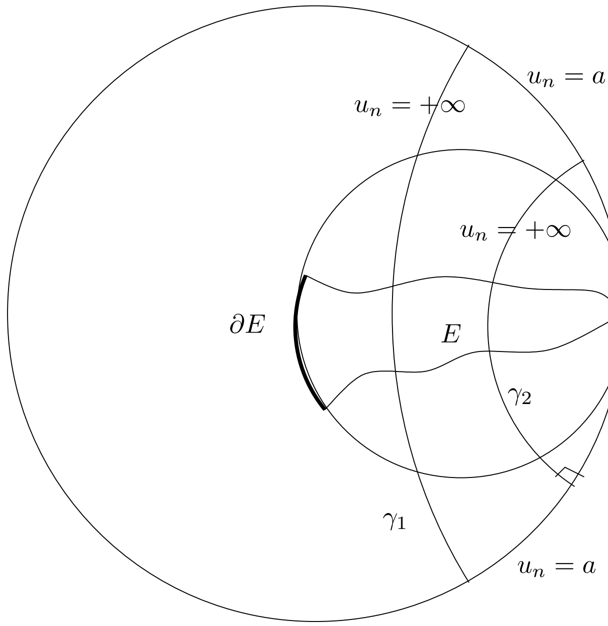

FIGURE 11. An annular end in $\mathcal{M}$

By a change of coordinates on $\mathcal{M}$, we can assume that the end A is in $\cup_{y\geq 1}\mathbb{T}(y)$ and $\partial A \subset \mathbb{T}(1)$. Let E be a connected component of the lift of A to $\mathbb{H}\times\mathbb{R}$. The boundary $\partial E \subset P := \{(x,y,t) \in \mathbb{R}^3; y = 1\}$ and E is transverse to P. There is (p,q) such that the curve ∂E is invariant by the isometry of $\mathbb{H}^2 \times \mathbb{R}$

$$\psi^p \circ T(h)^q : (x,y,t) \to (x + p\tau, y, t + qh).$$

We say that A and E are of type (p,q). The curve ∂A is a curve of the torus $\mathbb{T}(1)$. We prove in the following lemma that $(p,q) \neq (0,0)$

Lemma 4.2. *The end E is topologically a half-plane and ∂E is a non compact curve in P.*

Proof. Assume the contrary, and let E be a lifting of A to $\mathbb{H}\times\mathbb{R}$, E an immersed annulus in $\{y \geq 1\}$, $\partial E \subset H = \{y = 1\}$. We know the coordinate y is a proper function on E.

Denote by $\Pi : \mathbb{H}\times\mathbb{R} \to \mathbb{H} = \mathbb{H}\times\{0\}$, the vertical projection. Let γ_1, γ_2 be disjoint geodesics of $\mathbb{H}$, disjoint from $\Pi(\partial E)$, and that separate $\Pi(\partial E)$ from the point at infinity of $\mathbb{H}$; see figure 11 (the set $\Pi(\partial E)$ is compact). Let $\Omega \subset \mathbb{H}$ be the domain of $\mathbb{H}$ bounded by $\gamma_1 \cup \gamma_2$, so $\Pi(\partial E) \cap \Omega = \emptyset$.

For $a \in \mathbb{R}$, solve the Dirichlet problem on Ω to find a minimal graph over Ω, with asymptotic values $+\infty$ on $\gamma_1 \cup \gamma_2$, and a on $\partial_\infty(\Omega)$ (see Collin-Rosenberg [2]).

By varying a we obtain a first point of contact of the graph with E; a contradiction. $\square$

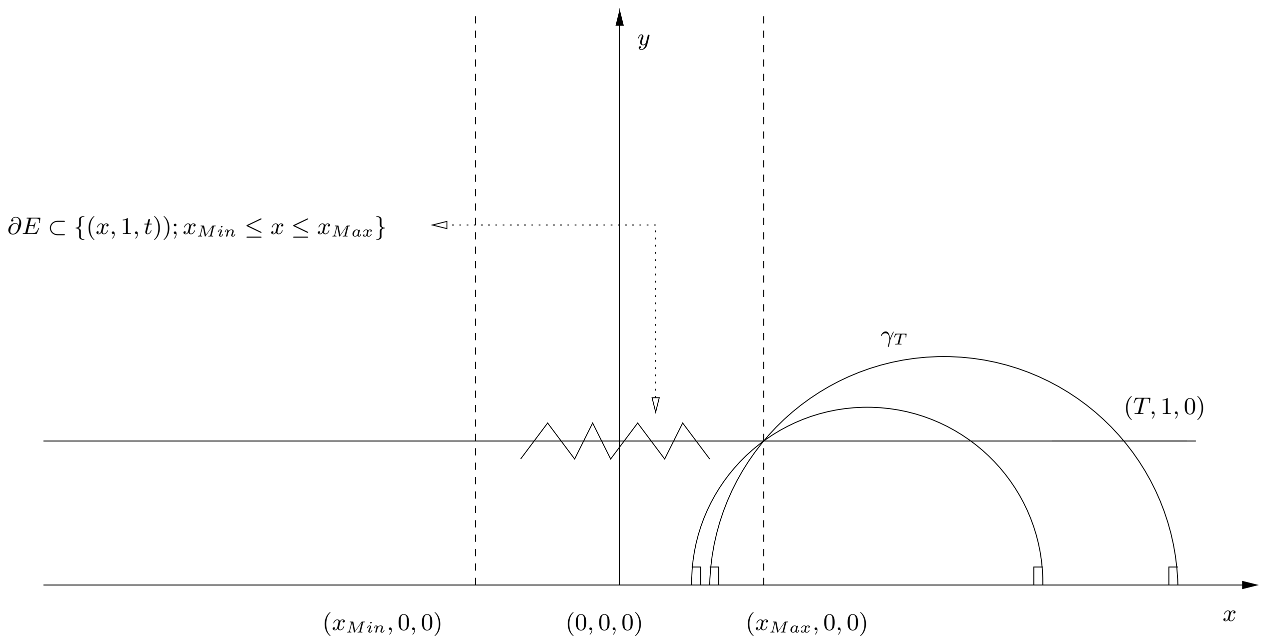

FIGURE 12. An annular end in $\mathcal{M}$

Now we prove that an end E of type (p,q), $(p,q) \neq (0,0)$ is trapped between two ends of type $E_{(p,q)}$:

Theorem 4.3. (The Trapping Theorem) *Let $A \subset \Sigma$ be a properly immersed end in $\mathcal{M}$ with $\partial A \subset \mathbb{T}(1)$ and A transverse to $\mathbb{T}(1)$. If ∂A, is a curve of type (p,q) in $\mathbb{T}(1)$, then A is contained in a slab on $\mathcal{M}$ bounded by two standard ends $A_{(p,q)}$.*

Proof. We use the model of $\mathcal{M} = (H \times \mathbb{R})/[\psi, T(h)]$ and a connected component E of a lifting of A in $\mathbb{H}^2 \times \mathbb{R}$. We prove that E is contained in a slab bounded by two half-planes $E_{(p,q)}$ in $H \times \mathbb{R}$ where $H = \{(x,y) \in \mathbb{R}^2; y \geq 1\}$.

Case $(p,q) = (0,q)$. First we begin with the case where the curve ∂A is of type $(0,q)$. This means that the boundary ∂E is a periodic curve invariant by vertical translation $T(h)^q = T(qh)$. From this invariance of ∂E, we know there exists x_{Min} and x_{Max} such that $\partial E \subset \{(x,1,t); x_{Min} \leq x \leq x_{Max}\}$.

Let $Q = \{(x,y) \in \mathbb{R}^2;\ x \geq x_{\text{Max}} \text{ and } y > 0\}$. Foliate Q by the geodesics γ_T whose end-points at infinity are $(x_{Max},0)$ and $(T,0)$; $T > x_{Max}$. For $|T - x_{Max}| < 2$, $\gamma_T \cap \{y \geq 1\} = \emptyset$. Define $S_T = \gamma_T \times \mathbb{R}$; so $S_T \cap E = \emptyset$ for $|T - x_{Max}| < 2$.

Now let T increase to ∞, so S_T converges to $\{(x_{Max},y); y > 0\} \times \mathbb{R}$. Since E is periodic, S_T must be disjoint from E for all $T > 1$; otherwise there would be a first point of contact (i.e. the two surfaces cannot have a first contact point at infinity), contradicting the maximum principle (see figure 12)

The same argument using x_{Min} shows E is trapped between two standard ends of type $E_{(0,1)}$.

Case $(p,q)=(p,0)$ Now ∂E is a curve invariant by $\psi^p(x,y,t)=(x+p\tau,y,t)$. Let t_{Min} and t_{Max} satisfy:

$$\partial E\subset\{(x,1,t);t_{Min}\leq t\leq t_{Max}\}.$$

Translate the barriers of lemma 3.2, see figure 8,

$$S^0_T(t):=\{(u,T\sin v,v+t)\in\mathbb{R}^3;u\in\mathbb{R},v\in[0,\pi]\}\text{ with }t\geq t_{\mathrm{Max}}.$$

For $|T|\leq 1$, we have $S^0_T(t)\cap(H\times\mathbb{R})=\emptyset$, and ∂E is below height $t=t_{Max}$. By lemma 3.2, the family $S^0_T(t_{Max})$ converges on compact sets to the horizontal section $t=t_{Max}$. For $t>t_{Max}$, t large, T_0 given, we have $S^0_{T_0}(t)\cap E=\emptyset$.
If $S^0_{T_0}(t_{Max})\cap E\neq\emptyset$, then since E is periodic, there would be a first t_1 such that $S^0_{T_0}(t_1)\cap E\neq\emptyset$, contradicting the maximum principle. Thus E is below $t=t_{Max}$. The same argument with

$$\tilde{S}^0_T(t):=\{(u,T\sin v,v-\pi+t)\in\mathbb{R}^3;u\in\mathbb{R},v\in[0,\pi]\}\text{ with }t\leq t_{\mathrm{Min}}$$

shows E is above $t=t_{Min}$. Thus E is trapped between two standard ends of type $E_{(p,0)}$.

Case $(p,q)\neq(0,q),(p,0)$. Now we use the family of barriers S^λ_T. ∂E is invariant by the isometry $\psi^p\circ T(h)^q:(x,y,t)\to(x+p\tau,y,t+qh)$ on $y=1$. Thus there exists $c_{\mathrm{Min}},c_{\mathrm{Max}}$ such that

$$\partial E\subset\{(x,1,t)\in\mathbb{R}^3;c_{\mathrm{Min}}\leq p\tau t-qhx\leq c_{\mathrm{Max}}\}.$$

We use S^λ_T of lemma 3.2 with $\lambda=\frac{qh}{p\tau}$; see figure 9.

$$S^\lambda_T(t):=\{(u,\alpha_T(v),v+\lambda u+t)\in\mathbb{R}^3;u\in\mathbb{R},v\in[0,v_0(T)]\}\text{ with }t\geq c_{\mathrm{Max}}/(p\tau)$$

For $T>1$ fixed, there is $t_0>c_{\mathrm{Max}}/(p\tau)$ large so that $S^\lambda_T(t_0)\cap E=\emptyset$. Decreasing t from t_0 to $c_{\mathrm{Max}}/(p\tau)$ we conclude (there is no first point of contact) that E is below $S^\lambda_T(c_{\mathrm{Max}}/(p\tau)$ for any T. Let $T\to\infty$; the $S^\lambda_T(c_{\mathrm{Max}}/(p\tau)$ converge to $\{(x,y,t)\in\mathbb{R}^3;p\tau t-qhx=c_{\mathrm{Max}}\}$, hence

$$E\subset\{(x,y,t)\in\mathbb{R}^3;p\tau t-qhx\leq c_{\mathrm{Max}}\}.$$

The same argument with

$$\tilde{S}^\lambda_T(t):=\{(u,\alpha_T(v),v+\lambda u-v_0(T)+t)\in\mathbb{R}^3;u\in\mathbb{R},v\in[0,v_0(T)]\}\text{ with }t\leq c_{\mathrm{Min}}/(p\tau)$$

shows that

$$E\subset\{(x,y,t)\in\mathbb{R}^3;p\tau t-qhx\geq c_{\mathrm{Min}}\},$$

which completes the proof of the theorem. □

5. The Dragging Lemma

Dragging Lemma 5.1. *Let $g : \Sigma \to N$ be a properly immersed minimal surface in a complete 3-manifold N. Let A be a compact surface (perhaps with boundary) and $f : A \times [0,1] \to N$ a $\mathcal{C}^1$-map such that $f(A \times \{t\}) = A(t)$ is a minimal immersion for $0 \le t \le 1$. If $\partial(A(t)) \cap g(\Sigma) = \emptyset$ for $0 \le t \le 1$ and $A(0) \cap g(\Sigma) \ne \emptyset$, then there is a $\mathcal{C}^1$ path $\gamma(t)$ in Σ, such that $g \circ \gamma(t) \in A(t) \cap g(\Sigma)$ for $0 \le t \le 1$. Moreover we can prescribe any initial value $g \circ \gamma(0) \in A(0) \cap g(\Sigma)$.*

Remark 5.2. *To obtain a $\gamma(t)$ satisfying the Dragging lemma that is continuous (not necessarily $\mathcal{C}^1$) it suffices to read the following proof up to (and including) Claim 1.*

Proof. When there is no chance of confusion we will identify in the following Σ and its image $g(\Sigma)$, $\gamma \subset \Sigma$ and $g \circ \gamma$ in $g(\Sigma) \subset N$. In particular when we consider embeddings of Σ there is no confusion.

Let $\Sigma(t) = g(\Sigma) \cap A(t)$ and $\Gamma(t) = f^{-1}(\Sigma(t))$, $0 \le t \le 1$ the pre-image in $A \times [0,1]$.

When $g : \Sigma \to N$ is an immersion, we consider $p_0 \in g(\Sigma) \cap A(0)$, and pre-images $z_0 \in g^{-1}(p_0)$ and $(q_0, 0) \in f^{-1}(p_0)$. We will obtain the arc $\gamma(t) \in \Sigma$ in a neighborhood of z_0 by a lift of an arc $\eta(t)$ in a neighborhood of $(q_0, 0)$ in $\Gamma([0,1])$ i.e. $g \circ \gamma(t) = f \circ \eta(t)$. We will extend the arc continuously by iterating the construction.

Since $\Gamma(t)$ represents the intersection of two compact minimal surfaces, we know $\Gamma(t)$ is a set of a finite number of compact analytic curves $\Gamma_1(t), ..., \Gamma_k(t)$. These curves $\Gamma_i(t)$ are analytic immersions of topological circles. By hypothesis, $\Gamma(t) \cap (\partial A \times [0,1]) = \emptyset$ for all t. The maximum principle assures that the immersed curves can not contain a small loop, nor an isolated point. Since $A(t)$ is compact and has bounded curvature, a small loop in $\Gamma(t)$ would bound a small disc D in Σ with boundary in A. Since A is locally a stable surface, we can consider a local foliation around the disc and find a contradicttion with the maximum principle. We say in the following that $\Gamma(t)$ does not contain small loops.

Claim 1: We will see that for each t with $\Gamma(t) \ne \emptyset$, $t < 1$ there is a $\delta(t) > 0$ such that if $(q, t) \in \Gamma(t)$, then there is a $\mathcal{C}^1$ arc $\eta(\tau)$ defined for $t \le \tau \le t + \delta(t)$ such that $\eta(t) = (q, t)$ and $\eta(\tau) \in \Gamma(\tau)$ for all τ (there may be values of t where $\gamma'(t) = 0$).

Since $\Gamma(0) \ne \emptyset$, this will show that the set of t for which $\eta(t)$ is defined is a non empty open set. This defines an arc $\gamma(\tau)$ as a lift of $f \circ \eta(\tau) \subset A(\tau)$ in a neighborhood of $\gamma(t) \in \Sigma$.

First suppose $(q, t) \in \Gamma(t)$ is a point where $A(t) = f(A \times \{t\})$ and $g(\Sigma)$ are transverse at $f(q, t)$. Let us consider the $\mathcal{C}^1$ immersions

$$F : A \times [0,1] \to N \times [0,1] \text{ with } F(q,t) = (f(q,t), t)$$

$$G : \Sigma \times [0,1] \to N \times [0,1] \text{ with } G(z,t) = (g(z), t).$$

Let $\hat{M} = F(A \times [0,1]) \cap G(\Sigma \times [0,1])$ and $M = F^{-1}(\hat{M})$. $F(A \times [0,1])$ and $G(\Sigma \times [0,1])$ are transverse at $p = F(q,t)$. Thus $\hat{M}$ is a 2-dimensional surface of $N \times [0,1]$ near p . We consider $X(t)$ a tangent vector field along $\Gamma(t)$ and $JX(t)$ an orthogonal

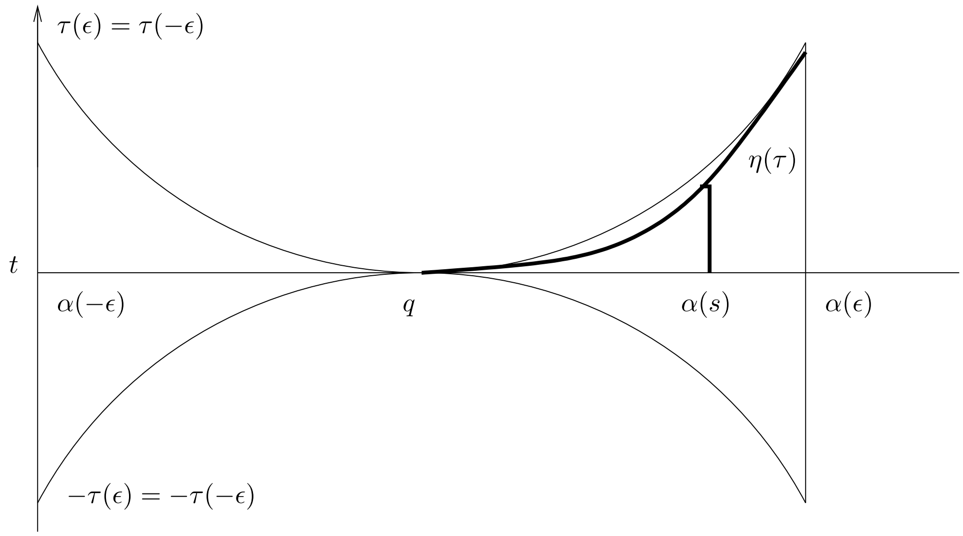

FIGURE 13. Neighborhood of a singular point

vector field to $X(t)$ in $T_{(q,t)}M$. If $\partial/\partial t \perp T_p\hat{M}$, then $T_p\hat{M} = T_{f(q,t)}A(t) = T_{f(q,t)}g(\Sigma)$ and (q,t) would be a non transverse point of intersection of $A(t)$ and $g(\Sigma)$. Thus $< JX(t), \partial/\partial t > \neq 0$ and we can find $\eta(\tau)$ a smooth path, defined for $\tau \in [t-\delta(q), t+\delta(q)]$ such that $\eta(t) = (q,t)$ and $\eta'(t) = JX(t)$ is transverse to $\Gamma(t)$ at (q,t).

By transversality and f being $\mathcal{C}^1$ in the variable t, we have a $\delta(q) > 0$ such that for $t-\delta(q) \leq \tau \leq t+\delta(q)$, $A(\tau)$ intersects $f \circ \eta(\tau)$ in a unique point and this point varies continuously with $t-\delta(q) \leq \tau \leq t+\delta(q)$. With a fixed initial point in Σ, a lift of $f \circ \eta(\tau)$, defines $\gamma(\tau) \in \Sigma$.

Again by transversality, we can find a neighborhood of (q,t) in $\Gamma(t)$ and a $\delta > 0$ so that the above path $\gamma(\tau)$ exists for $t-\delta \leq \tau \leq t+\delta$, through each point in the neighborhood of q. It suffices, to look for a local immersion of a neighborhood of 0 in T_pM into M, to obtain a $\mathcal{C}^1$ diffeomorphism $\psi : B(0) \subset T_pM \to M$. M has the structure of a $\mathcal{C}^1$ manifold in a neighborhood of points of transversality and this structure extends to $F^{-1}(M) \subset A \times [0,1]$.

We will find a $\delta > 0$ that works in a neighborhood of a singular point $(q,t) \in \Gamma(t)$, where there is a $z \in \Sigma$ such that $f(q,t) = g(z)$ and $T_{f(q,t)}A(t) = T_{g(z)}g(\Sigma)$. We consider singularities of $\Gamma(t)$ where $A(t)$ and $g(\Sigma)$ are tangent. Near a singularity $(q,t) \in \Gamma(t)$, $\Gamma(t)$ contains $2k$ analytic curves intersecting at q at equal angles, $k \geq 1$.

Let V be a neighborhood of q in A. The set $\Gamma(t) \cap V$ is $2k$ analytic curves. Let $\alpha :]-\epsilon, \epsilon[\to V \cap \Gamma(t)$ be a regular parametrization of one curve with $\alpha(0) = q$ and $\alpha(\pm\epsilon) \in \partial V$. By transversality as discussed in the previous paragraph $< JX(t), \partial/\partial t > \neq 0$ at $\alpha(s)$ for $s \neq 0$ and $JX(t)$ can be integrated as a curve on M for $t-\delta(s) \leq \tau \leq t+\delta(s)$. Here $\delta(s)$ is a $\mathcal{C}^1$ function which can be chosen increasing with $\delta(0) = \delta'(0) = 0$.

There exists a $\mathcal{C}^1$ diffeomorphism $\phi : \Omega = \{(s,\tau) \in \mathbb{R}^2; -\epsilon \leq s \leq \epsilon, t-\delta(s) \leq \tau \leq t+\delta(s)\} \to M$ such that $\phi(s,t) = \alpha(s)$ for $s \in]-\epsilon, \epsilon[$ and $\phi(s,\tau) \in \Gamma(\tau)$ for

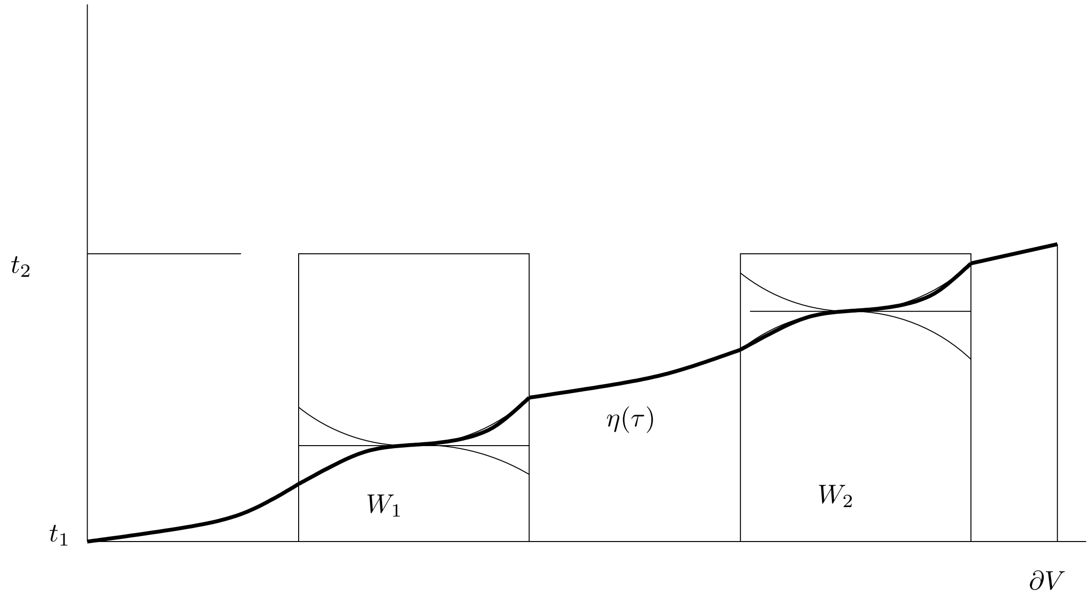

FIGURE 14. The curve $\eta(\tau)$ passing through several singularities.

$t-\delta(s) \leq \tau \leq t+\delta(s)$. We consider a function $\tau :]-\epsilon, \epsilon[\to \mathbb{R}$, such that $(s, \pm\tau(s)) \in \Omega$ and τ is increasing, $\tau(0) = \tau'(0) = 0$ and $\tau(\epsilon) = t + \delta(\epsilon)$, $\tau(-\epsilon) = t + \delta(-\epsilon)$.

Now we can construct a path $\eta(\tau) \in \Gamma(\tau)$ which joins (q, t) to a point in $\Gamma(t + \delta(\epsilon))$. The $\mathcal{C}^1$ arc $f \circ \eta(\tau), t \leq \tau \leq t + \delta(\epsilon)$ is locally parametrized by $\phi(s, \tau(s)), s \in]0, \epsilon[$ and continuously extends to $f(q, t)$ when $\tau \to t$. Each point $\alpha(s)$, can be connected $\mathcal{C}^1$, by the arc $\phi(s, \tau), t \leq \tau \leq \tau(s)$ from $\alpha(s)$ to $\phi(s, \tau(s))$, and next a subarc of $\eta(\tau)$ for $\tau(s) \leq \tau \leq t + \delta(\epsilon)$ (see figure 13). The constant $\delta(\epsilon)$ depends only on $\alpha(\epsilon) = q_1$, and we note $\delta(q_1) := \delta(\epsilon)$.

Now there are a finite number of arcs α in $V - (q)$, with end points q and a collection of $q_1, q_2, ...q_{2k}$. So one has a $0 < \delta$ with $\delta < \delta(q_i)$ that works in a neighborhood of q. The claim is proved.

To complete the proof of the Dragging Lemma, it suffices to prove that $\gamma(t)$ extends $\mathcal{C}^1$ for any value of $t \in [0, 1]$. Assume that there is a point t_0 such that the arc $\gamma(t)$ is defined in a $\mathcal{C}^1$ manner for $t < t_0$. By compactness of A, the arc accumulates at a point $(q, t_0) \in \Gamma(t_0)$. Remark that the structure of M along $\Gamma(t_0)$ gives easily the existence of a continuous extension to t_0. To ensure a $\mathcal{C}^1$ path through t_0, we need a more careful analysis at (q, t_0).

Claim 2: Suppose the path $\gamma(t)$ satisfies the conditions of the Dragging lemma for $0 \leq t \leq t_0 < 1$. Then $\gamma(t)$ can be extended to $0 < t < t_0 + \delta$, to be $\mathcal{C}^1$ and satisfy the conditions of the Dragging lemma, for some $\delta > 0$.

If (q, t_0) is a transversal point, M has a structure of a manifold and if $t_0 - \delta(t_0) < t_1 < t_0$ and $\eta(t_1) = (q_1, t_1)$ is in a neighborhood of (q, t_0), we can find a $\mathcal{C}^1$ arc that joins $\eta(t_1)$ to $(q, t_0) \in \Gamma(t_0)$. Next we extend the arc for $t_0 \leq t \leq t_0 + \delta(t_0)$.

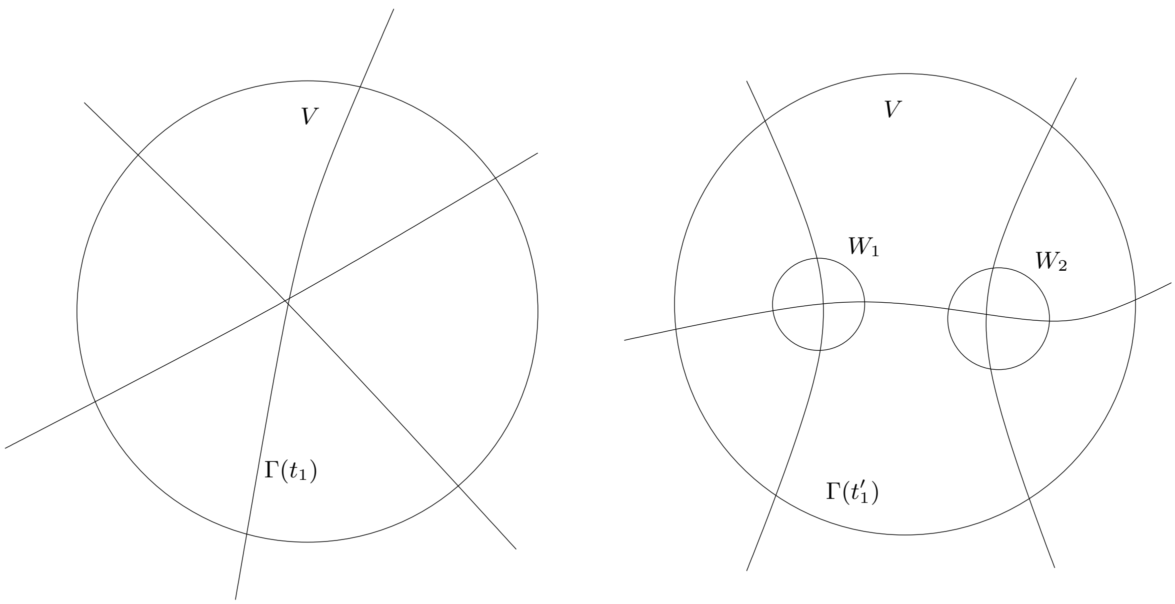

FIGURE 15. Left: The curve $\Gamma(t_1)$-Right: The curve $\Gamma(t_1')$.

If (q, t_0) is a singular point, we consider a neighborhood $V \subset A$ of q and $\Gamma(t_0)$ intersects ∂V in $2k$ transversal points $q_1, ..., q_{2k}$. We consider $V \times [t_1, t_0]$ with $t_0 - \delta(t_0) < t_1 < t_0$. By transversality at $(q_1, t_0), ..., (q_{2k}, t_0)$, the analytic set $\Gamma(t_1)$ intersects ∂V in $2k$ points and V in k analytical arcs $\alpha_1, ..., \alpha_k$. We suppose that $\eta(t_1) \in \alpha_1 \subset V \times \{t_1\}$. We construct below a monotonous $\mathcal{C}^1$ arc from $\eta(t_1)$ to a point $(\hat{q}, t_2)$ on $\partial V \times \{t_2\}$ for some $t_1 < t_2 < t_0$ and by transversality an arc from $(\hat{q}, t_2)$ to a point $(q', t_0) \in \partial V \times \{t_0\}$, using the fact that $t_0 - \delta(t_0) < t_2$. Next we can extend the arc in a $\mathcal{C}^1$ manner from (q', t_0) to some point in $\Gamma(t_0 + \delta(t_0))$.

We consider $(\tilde{q}_1, t_1), ..., (\tilde{q}_\ell, t_1)$ singular points of $\Gamma(t_1) \cap V \times \{t_1\}$ and we denote by $W_1, ..., W_\ell$ neighborhoods of $\tilde{q}_1, ..., \tilde{q}_\ell$ in $A \cap V$. The arc α_1 cannot have double points in V without creating small loops. Hence α_1 passes through each $W_1, ..., W_\ell$ at most one time, before joining a point of ∂V (We can restrict V in such a way that there are no small loops in V).

First we assume that there is t_2 such that for any $t \in [t_1, t_2]$, the curve $\Gamma(t)$ has exactly one isolated singularity in each neighborhood $W_i \times \{t\}$ with the same type as $\tilde{q}_i \in \Gamma(t_1)$ $(i = 1, ..., \ell)$ and $t_2 < t_1 + \delta(t_1)$. If we parametrize $\alpha_1 : [s_0, s_{2\ell+1}] \to \Gamma(t_1)$, we can find $s_1, ..., s_{2\ell}$ such that $\alpha_1(s_{2k-1}), \alpha_1(s_{2k}) \in \partial W_k$ and $I_k = [s_{2k-2}, s_{2k-1}]$ are intervals parametrizing transversal points in $\Gamma(t_1)$.

The manifold structure of M gives an immersion $\psi_j : I_j \times [t_1, t_1 + \delta] \to M$, $t_1 + \delta < t_2$ and $j = 1, ..., \ell + 1$. In the construction of η up to t_1, the singular points are isolated; then we can assume $\eta(t_1)$ is a regular point of $\Gamma(t_1)$, hence is contained in an $\alpha_1(I_j)$. We construct the beginning of the arc $\eta(\tau)$ as the graph parametrized by $\phi_j(s, \tau(s))$ with τ an increasing function from t_1 to $t_1 + \delta/n$ as s varies from $\hat{s} \in I_j$, corresponding

to the initial point $\eta(t_1) = \alpha_1(\hat{s})$, to s_{2j-1}. Next we pass through the singularity $(\tilde{q}_j, t_1 + 2\delta/n)$ by constructing an arc wich joins the point $\phi_j(s_{2j-1}, t_1 + \delta/n) \in \Gamma(t_1 + \delta/n) \cap \partial W_j$ to the point $\phi_{j+1}(s_{2j}, t_1 + 3\delta/n) \in \Gamma(t_1 + 3\delta/n) \cap \partial W_j$ (see figure 14). For a suitable value of n we can iterate this construction, passing through the singularities $\tilde{q}_j, \tilde{q}_{j+1}...$, until we join a point $(\hat{q}, t_2)$ of $\partial V \times \{t_2\}$ and then we extend the arc up to t_0 by transversality outside V.

Now we look for this interval $[t_1, t_2]$. Let $t_1 < t_1' < t_0$ and $\Gamma(t_1')$ have several singularities in some neighborhood W_k, or a unique singularity of index less the one of the $\tilde{q}_k$. We consider in this W_k a finite collection of neighborhoods of isolated singularities $W'_{k,1}, ...W'_{k,\ell'}$. We observe, by transversality that there are the same number of components of $\Gamma(t_1)$ and $\Gamma(t_1')$ in W_k (see figure 15). Hence each $W'_{k,j}$ contains a number of components of $\Gamma(t_1')$ strictly less than the number of components of $\Gamma(t_1)$ in W_k. The index of the singularity is strictly decreasing along this procedure. We can iterate this analysis up to a point where each singularity can not be reduced to a simple one. This gives the interval $[t_1, t_2]$.

□

6. Compact minimal annuli

We now introduce the compact stable horizontal minimal annulus F_0 bounded by circles in vertical planes $P(c)$ and $P(-c)$ where $P(c) = \{(x, y, t) \in \mathbb{R}^3; y > 0 \text{ and } x = c\}$. We also foliate a tubular neighborhood $\mathrm{Tub}(F_0)$ of F_0 by compact minimal annuli F_s, $-1 \leq s \leq 1$ and certain small balls B_ρ containing horizontal minimal annuli $\mathcal{C}_\ell$.

We will now construct the stable compact annulus F_0. Let η be a circle of radius one in the vertical plane $P(c)$, centered at $(c, y, 0)$. The metric induced on $P(c)$ is euclidean so circles make sense. As $y \to \infty$, $\mathrm{dist}(\eta, P(0)) \to 0$. The disk of least area in $\mathbb{H}^2 \times \mathbb{R}$ bounded by η is the disc in $P(c)$ bounded by η (by the maximum principle). The area of this disk does not depend y. So for y large, there is a compact annulus in $\mathbb{H}^2 \times \mathbb{R}$ with one boundary in $P(0)$ and the other boundary η, whose area is less that the area of the disk in $P(c)$ bounded by η. Assume y is large. then by the Douglas criterium there is a least area annulus F_+ having one boundary η and the other in $P(0)$. Since F_+ has least area w.r.t. this boundary condition, $\partial F_+ \cap P(0)$ is orthogonal to $P(0)$. Hence F_0, the symmetry of F_+ through $P(0)$, union F_+, is a smooth compact minimal annulus orthogonal to $P(0)$ and $F_+ \cap P(0)$ is convex. The normal vector along this curve takes on all directions in the plane $P(0)$. Let σ be symmetry through $P(0)$, $\eta_- = \sigma(\eta)$, $F_- = \sigma(F_+)$. Observe that F_0 has least area with boundary $\eta \cup \eta_-$. For if B is an annulus with $\partial B = \eta \cup \eta_-$, write $B = B_+ \cup B_-$ where $B_+ = \{(x, y, t) \in \mathbb{R}^3; y > 0 \text{ and } 0 \leq x \leq c\} \cap B$, and $B_- = \{(x, y, t) \in \mathbb{R}^3; y > 0 \text{ and } -c \leq x \leq 0\} \cap B$. We know that the $\mathrm{Area}(B_+) = |B_+| \geq \frac{|F_0|}{2}$ and $|B_-| \geq \frac{|F_0|}{2}$ so $|B| \geq |F_0|$. Thus F_0 is a stable annulus as desired. Let γ_1 be the geodesic joining $(c, y, 0)$ to $(-c, y, 0)$. We assume y large so that $\gamma_1 \cap F_0 = \emptyset$.

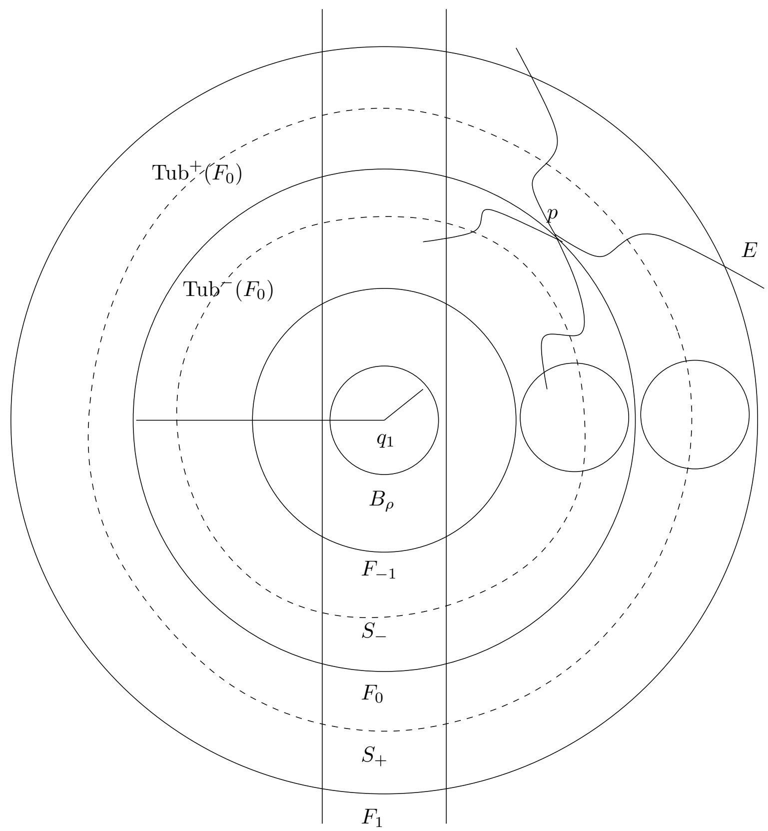

FIGURE 16.

Let $\eta(s)$ be equidistant circles of η in $P(c)$, for $|s|$ small, $\eta(0) = \eta$. Let $\eta_-(s) = \sigma(\eta(s))$ be equidistant circles in $P(-c)$, $\eta_-(0) = \eta_-$. Since F_0 is strictly stable, there is a $\delta > 0$ so that for $|s| \leq \delta$, there is a foliation of a tubular neighborhood $\mathrm{Tub}(F_0)$, of F_0 by compact minimal annuli $F(s)$, with $\partial F(s) = \eta(s) \cup \eta_-(s)$. choose δ sufficiently small so that

$$\mathrm{dist}(\mathrm{Tub}(F_0), \gamma_1) > 0.$$

Let $\mathrm{Slab}(c) = \{(x, y, t) \in \mathbb{R}^3; y > 0 \text{ and } |x| \leq c\}$. We denote by F_0^- the bounded component of $\mathrm{Slab}(c) - F_0$, and by F_0^+ the other component of $\mathrm{Slab}(c) - F_0$. The annuli F_s are inside F_0^- for $s \in [-1, 0]$ and inside F_0^+ for $s \in [0, 1]$.

We consider $\mathrm{Tub}^-(F_0) = \cup_{s\in[-1,0]} F_s$ and $\mathrm{Tub}^+(F_0) = \cup_{s\in[0,1]} F_s$; domains of $\mathbb{H}^2 \times \mathbb{R}$.

We consider the curves $S_+ = P(0) \cap F_{1/2}$ and $S_- = P(0) \cap F_{-1/2}$. There exists a constant $\rho > 0$, such that for any q of S_+(or S_-) the geodesic ball $B_\rho(q)$ of geodesic radius ρ centered at q is contained in $\mathrm{Tub}^+(F_0)$ (resp. $\mathrm{Tub}^-(F_0)$).

We can find $\ell > 0$ such that any geodesic ball of radius ρ centered at q contains a small compact minimal annulus $\mathcal{C}_\ell$ bounded by two geodesic circles contained in $P(\ell) \cap B_\rho$ and $P(-\ell) \cap B_\rho$. We say in the following that $\mathcal{C}_\ell$ is centered at $q \in S_+ \cup S_-$. We denote by q_1 the point $\gamma_1 \cap P(0)$; see figure 17.

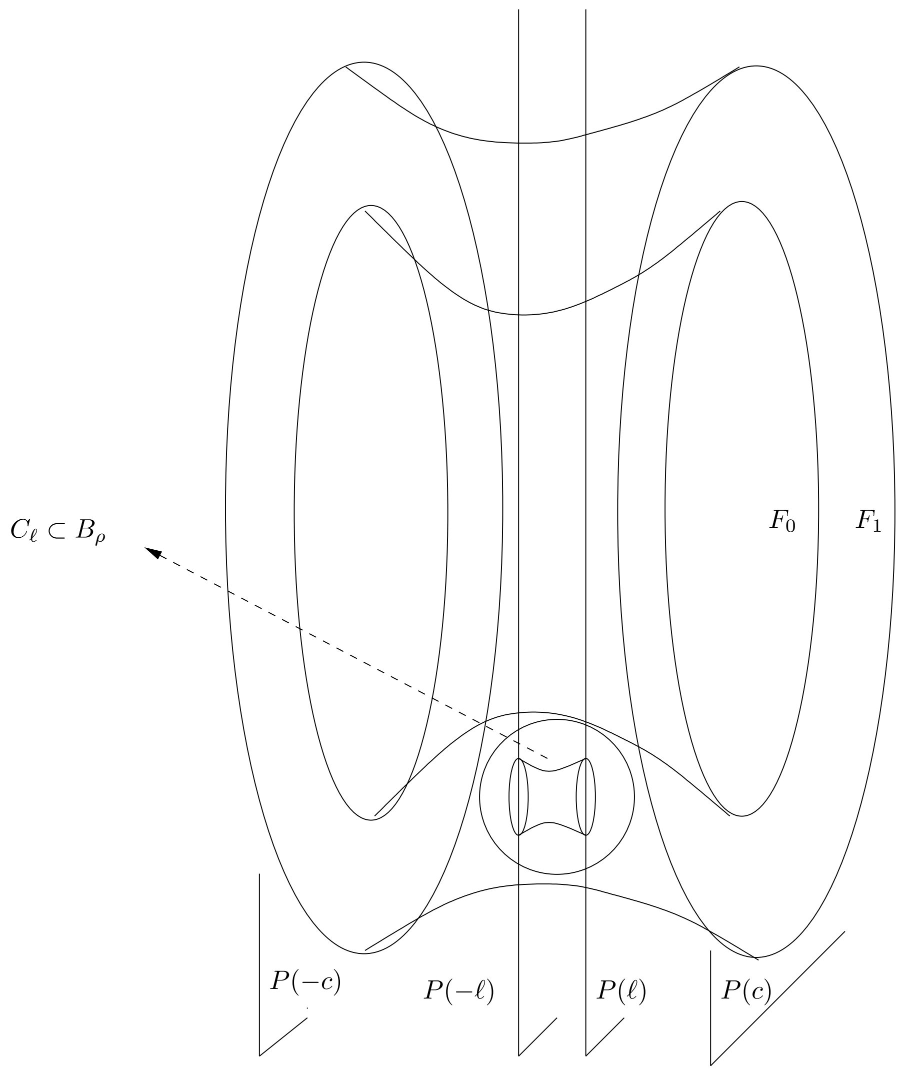

FIGURE 17.

In summary, we fix $\rho > 0$ such that

(1) $3\rho < \mathrm{dist}(F_0, \gamma_1)$,
(2) $B_{2\rho}(q_1) \cap \mathrm{Tub}^-(F_0) = \emptyset$
(3) $B_\rho(q) \subset \mathrm{Tub}^-(F_0)$ for any $q \in S_-$,
(4) $B_\rho(q) \subset \mathrm{Tub}^+(F_0)$ for any $q \in S_+$,

Now we clearly have the following:

Claim: Any continuous curve γ in the interior of $\mathrm{Tub}^+(F_0) \cap \mathrm{Slab}(\ell)$ (or $\mathrm{Tub}^-(F_0) \cap \mathrm{Slab}(\ell)$) joining F_0 to F_1 (or F_{-1}) intersects a compact annulus of the family $C_\ell(q) \subset \mathrm{Tub}^+(F_0)$(resp. $\mathrm{Tub}^-(F_0)$) for some point $q \in S_+$ (reps. $q \in S_-$). The next proposition gives at least two components of Σ in F_0^- when F_0 is tangent to Σ at some point.

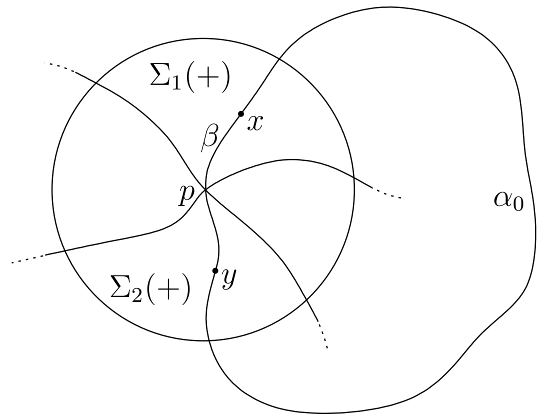

FIGURE 18.

Proposition 6.1. *Let Σ be a properly immersed minimal half-plane in* $\mathrm{Slab}(\ell)$*. Suppose Σ is tangent to F_0 at p and $\partial\Sigma \cap F_0 = \emptyset$. Then there are at least two connected components of Σ in F_0^-. More precisely if $\Sigma_1(-)$ and $\Sigma_2(-)$ are distinct local components of Σ in F_0^- then $\Sigma_1(-)$ and $\Sigma_2(-)$ are in distinct components of $\Sigma \cap F_0^-$.*

Proof. If not we can find a path α_0 in $\Sigma \cap F_0^-$, joining a point $x \in \Sigma_1(-)$ and $y \in \Sigma_2(-)$. Then join x to y by a local path β_0 in Σ going through p, but $\beta_0 \subset F_0^-$ except at p (see figure 18). Let $\Gamma = \alpha_0 \cup \beta_0 \subset F_0^-$. Since Σ is a half-plane, Γ bounds a disk D in Σ. By construction D contains points in the interior of F_0^+.

Hence there is a compact component of D in F_0^+ with boundary in F_0. By the maximum principle $D \cap F_t \neq \emptyset$, $0 \leq t \leq 1$ and there is at least one point p_1 of $D \cap F_1$. Using compact annuli $\mathcal{C}_\ell$ inside $\mathrm{Tub}^+(F_0)$, we can find an annulus $\mathcal{C}_\ell(q)$ which intersects D (by the claim). Now translate this catenoid in the interior F_0^+ to a point outside the convex hull of F_0. Apply the Dragging lemma to obtain points of D outside the convex hull. This contradicts the maximum principle. □

7. A FAMILY OF GRAPH BARRIERS

In this section we study a one parameter family of surfaces Σ_n graphs on a sequence of domains Ω_n of $\mathbb{H}^2$bounded by two geodesics. In the unit disk model of $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$, we consider two geodesics γ_n and γ_{-n} passing through the points $(-1 + 1/n, 0)$ and $(1 - 1/n, 0))$ and both orthogonal to $\{y = 0\}$. We consider the domain Ω_n bounded by γ_n and γ_{-n} (see figure 20, Left). We solve the minimal graph equation for a function $u_n : \Omega_n \to \mathbb{R}$ with $u_n = +\infty$ on $\gamma_n \cup \gamma_{-n}$ and $u_n = 0$ on $\partial_\infty \Omega_n$, the boundary at infinity of Ω_n.

The graph u_n has a line of curvature Γ_n over the geodesic $\gamma_0 = \{(x, y) \in \mathbb{H}^2; x = 0\}$. The following proposition describes the limit of the graphs Σ_n when $n \to \infty$.

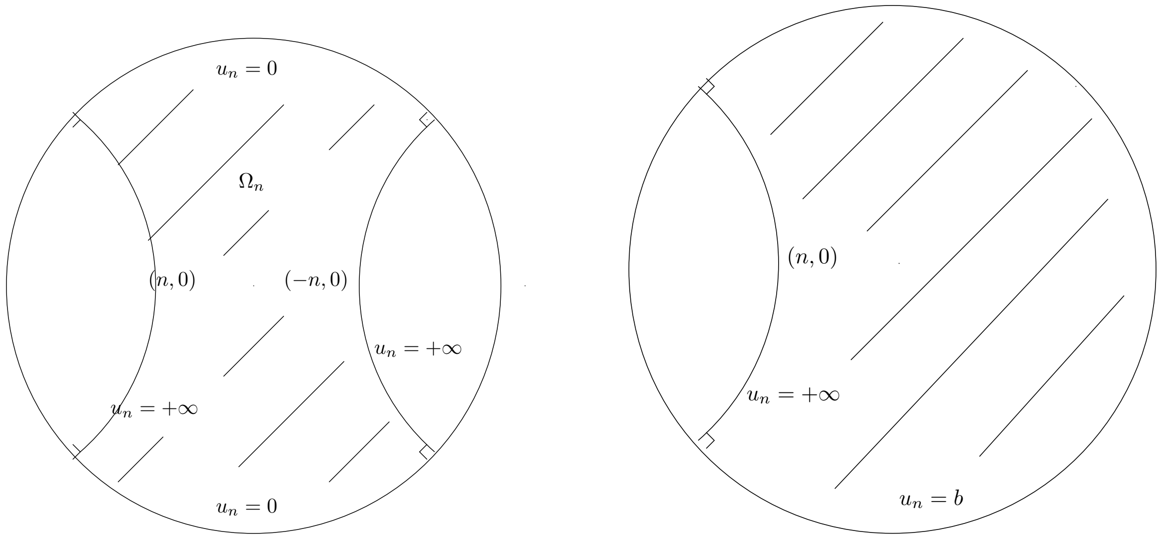

FIGURE 19. Left: Domain Ω_n of function u_n. Right: Domain of functions v_n.

Proposition 7.1. *The sequence of solutions of the minimal graph equation in the sequence of domains Ω_n with boundary data $u_n = +\infty$ on $\gamma_n \cup \gamma_{-n}$ and $u_n = 0$ on $\partial_\infty \Omega_n$, converge uniformly to the horizontal section $\mathbb{H}^2 \times \{0\}$.*

Proof. The sequence of domains Ω_n is an increasing sequence in $\mathbb{H}^2$; $\Omega_n \subset \Omega_{n+1}$. The maximun principle assures that the sequence is decreasing with $0 \leq u_{n+1}(q) \leq u_n(q)$ for any $q \in \Omega_n$. Hence the sequence of graphs Σ_n converges to an entire graph of a function $u_0 : \mathbb{H}^2 \to \mathbb{R}$. We will prove that $u_0 \equiv 0$.

It suffices to prove that $u_0 = 0$ on the geodesic γ_0. If not we can assume that $\sup_{\gamma_0} u_n = a_n \geq b > 0$ and there is $p \in \gamma_0$ such that $u_0(p) = b$.

This point p exists because u_0 takes value 0 at infinity of γ_0. This comes from the fact that (u_n) is a decreasing sequence hence $u_n = 0$ at infinity of Ω_n for any $n \in \mathbb{N}$.

We consider the sequence of minimal surfaces $\tilde{\Sigma}_n$ graphs of a function v_n on a domain V_n bounded by γ_n, with boundary data $v_n = +\infty$ on γ_n and $v_n = b$ on $\partial_\infty V_n$. This family of graphs is well known and Mazet, Rodriguez and Rosenberg proved in [8] that the sequence v_n converges uniformly to $v_0 \equiv b$, when $n \to \infty$.

We restrict the function $v_n : V_n \to \mathbb{R}$ to the domain W_n bounded by the geodesic γ_n and the geodesic γ_0. On W_n we claim that the maximum principle applies to show that $v_n \geq u_0$. To see this its suffices to check for the inequality on the boundary of the domain W_n. On γ_n, the function $v_n = +\infty > u_0$ and on γ_0, we have $v_n \geq b \geq u_0$. At the boundary at infinity $\partial_\infty V_n$ we have $v_n = b > 0 = u_0$.

Now let $n \to \infty$ to show that $v_n \to v_0 = b$ and $v_0 \geq u_0$. This proves by symmetry that the function $u_0 \leq b$. The point p of γ_0 where $u_0(p) = b$ is an interior maximum

point of the function, hence $u_0 = b$. This contradicts the fact that u_0 take the value 0 at the boundary at infinity of $\mathbb{H}^2$. $\square$

We consider an end $E_{(1,0)}(c)$ contained in a slab $S = \{(x, y, t) \in \mathbb{R}^3; y \geq 1 \text{ and } -c_1 \leq t \leq c_1\}$ and we use the proposition 7.1 to obtain $\mathcal{C}^0$ convergence:

Proposition 7.2. *An end A of type $(p, 0)$ of a properly immersed minimal surface Σ in $M \times \mathbb{S}^1$ has third coordinate which has a limit at infinity i.e. that A converges in the $\mathcal{C}^0$ norm to $A_{(p,0)}$ at height $a \in]-c_1, c_1[$.*

Remark 7.3. *We will prove in the next proposition that A converges uniformly in the $\mathcal{C}^2$ norm to $A_{(p,0)}$ i.e; A is a graph converging uniformly to the cusp $A_{(p,0)}$ at height $\{t = a\}$.*

Proof. A covering E of A is contained in a slab bounded by $E_{(1,0)}(-c_1)$ and $E_{(1,0)}(c_1)$. We study the intersection of E with the level section $E_{(1,0)}(c) = \{(x, y, t) \in \mathbb{R}^3; y \geq 1 \text{ and } t = c\}$ with $c \in]-c_1, +c_1[$. If Γ is a compact component of $A \cap E_{(1,0)}(c)$ then $\Gamma \cap \partial A \neq \emptyset$. Otherwise Γ bounds a disc D or a subend A_0. In both cases, the maximum principle of proposition 7.1 applies and $A = A_{(p,0)}$ i.e. A is a flat standard end of height c.

Varying the value $c \in]-c_1, +c_1[$, there is a value $a \in]-c_1, +c_1[$, such that the intersection $A \cap E_{(1,0)}(c)$ has a non compact component denoted by Γ. By proposition 7.1, $\Gamma \cap \partial E \neq \emptyset$. In the lift E of A, we consider two lifts of Γ denoted by Γ_1 and $\Gamma_2 := \psi \circ \Gamma_1$. The curves Γ_1, Γ_2 and a compact arc $\Gamma_3 \subset \partial E$ bound a fundamental domain, (see figure 19).

We consider a graph obtained by a translation σ_n of Σ_n of section 7 such that the geodesic γ_{-n} translates to a fixed geodesic $\bar{\gamma}$ which does not intersect $\Gamma_3 \subset \partial E$ and Σ_n is above E with boundary data $u_n = a$ at the boundary at infinity. The graph Σ_n has a line of curvature which is a graph over the translation of the geodesic γ_0 denoted by $\sigma_n \circ \gamma_0$ and the boundary curve γ_n is sent to $\sigma_n \circ \gamma_n$. We remark that $\sigma_n \circ \gamma_0$ is a distance n from $\bar{\gamma}$ and $\sigma_n \circ \gamma_n$ at a distance $2n$ from $\bar{\gamma}$.

We let $n \to \infty$ and fix the geodesic $\bar{\gamma}$, using the horizontal isometry σ_n. These graphs are conjugate to the graphs of the sequence u_n of proposition 7.1. We know that the graph over $\sigma_n \circ \gamma_0$ is converging to the height a, hence we see that the end A cannot have a point above the height a at infinity. We do the same with a symmetric graph with value $-\infty$ on the geodesic γ_{-n} and γ_{+n}. This proves that the end A is trapped between two graphs which have third coordinate going to the same value a. Hence the end A converges in the $\mathcal{C}^2$ norm to a cusp end $t = a$. $\square$

8. Proof of the theorem in $M \times \mathbb{S}^1$.

The surface Σ is properly immersed. By lemma 4.1, 4.2 and theorem 4.3, each end A lifts to E a half-plane trapped between two standard ends $E_{(p,q)}$. First we prove the theorem for an end of type $(0, p)$ and then we adapt the arguments to the general case.

Ends of type $(0,p)$**; the vertical case.** Since E is trapped between two vertical planes and the distance between two vertical planes tends to zero as $y \to \infty$, we can assume $E \subset \mathrm{Slab}(\ell/2)$ where $\mathrm{Slab}(c) = \{(x,y,t) \in \mathbb{R}^3; y > 0 \text{ and } |t| \leq c\}$, and $\partial E \subset \{y = y_0\}$. We will prove that when $p \in E$, and $y(p) > 3 + y_0$, then the killing field $\frac{\partial}{\partial x}$ is transverse to E at p. Then this sub-end of E is stable hence has bounded curvature. We will prove later that this gives the theorem in this case.

Suppose on the contrary that some p, with $y(p) > 3+y_0$, has $\frac{\partial}{\partial x}|_p$ in T_pE. The annulus F_0 meets $P(0)$ orthogonally and the normal vector to this curve of intersection is in the plane $P(0)$ and takes all directions in this plane as one goes once around the curve.

Since $\frac{\partial}{\partial x}|_p \in T_pE$, the normal vector to E at p is in the plane $P(x = x(p))$. Thus we can translate F_0 to p (call F_0 this translated F_0) to be tangent to E at p. By a translation of less than $\ell/2$ we can assume $x(p) = 0$, so now $E \subset \mathrm{Slab}(\ell)$. Recall that γ_1 is the geodesic joining the centers of the boundary circles of F_0 and $q_1 = \gamma_1 \cap P(0)$. Write $q_1 = (0, y_1, 0)$ with $y_1 > 2 + y_0$.

The convex hull of the foliation of $\mathrm{Tub}^-(F_0) \cup \mathrm{Tub}^+(F_0) \cup F_0$ has y coordinate at least the minimum of the y-coordinate of the boundary circles of $F(t)$ i.e. $y \geq y_0 + 1/2$ on the convex hull. Now we proved in the section 6, that this foliation contains a family of geodesic balls $B_\rho(q)$ of radius $\rho > 0$ centered at points $q \in S_+ \cup S_-$. We choose this constant ρ such that

$$3\rho < \mathrm{dist}(F_0, \gamma_1) \text{ and } 4\rho < 1.$$

Each such geodesic ball $B(q)$ contains a compact annulus $\mathcal{C}_\ell(q)$ bounded by geodesic circles of radius δ contained in $P(\ell)$ and $P(-\ell)$.

Step 1: Construction of arcs on E. We know by proposition 6.1, that there are at least two connected components Σ_1, Σ_2 of $E - F_0$ that have p in their closure, and $\Sigma_1, \Sigma_2 \subset F_0^-$. Clearly, by the maximum principle, each of Σ_1, Σ_2 intersects each of the catenoids in the local foliation F_s about F_0 in F_0^-. In particular there is a $\tilde{q} \in S_-$ such that $\mathcal{C}_\ell(\tilde{q}) \cap \Sigma_1 \neq \emptyset$. Now translate $\mathcal{C}_\ell(\tilde{q})$ along the geodesic joining $\tilde{q}$ to q_1 and apply the Dragging lemma to obtain a point $p_1 \in \Sigma_1 \cap \mathcal{C}_\ell(q_1) \subset B_\rho(q_1)$.

The same argument gives a point $p_2 \in \Sigma_2 \cap \mathcal{C}_\ell(q_1)$. Recall that p_1 and p_2 can not be joined by an arc in $E \cap F_0^-$ (we will use this later). Now we construct a loop μ in E.

For a value $k_0 \in \mathbb{N}$ which will be defined in step 2, we consider Γ_+ to be the euclidean segment joining $q_1 = (0, y_1, 0)$ to $(0, y_1, k_0 h + 2\rho)$, together with the segment joining $(0, y_1, k_0 h + 2\rho)$ to $z = (0, y_0, k_0 h + 2\rho)$. We will connect the point p_1 and p_2 by an arc in E which stays in a tubular neighborhood of $\Gamma_+ \cup \partial E$. We note by $\mathrm{Tub}_\rho(\Gamma_+)$ the tubular neighborhood of geodesic radius ρ along Γ_+. We parametrize the curve Γ_+ in a piecewise $\mathcal{C}^1$-monotone manner by $q(\bar{t}), 0 \leq \bar{t} \leq 1$ and we move $B_\rho(q_1)$ along $q(\bar{t})$, from q_1 to $z = (0, y_0, k_0 h + 2\rho)$, by $B_\rho(q(\bar{t}))$. Each ball $B_\rho(q(\bar{t}))$, $q \in \Gamma_+$ contains the catenoid $\mathcal{C}_\ell(q(\bar{t}))$ and the Dragging lemma then gives two continuous paths $\sigma_1^+(\bar{t}), \sigma_2^+(\bar{t})$ starting at p_1,p_2 respectively such that $\sigma_i^+(\bar{t}) \in E$ for $0 \leq \bar{t} \leq 1$.

We apply the Dragging lemma up to the value $q(1) = z$ and $\sigma_i^+(1) \in B_\rho(z)$ for $i = 1, 2$. Since $\tilde{p}_1 = \sigma_1^+(1)$ and $\tilde{p}_2 = \sigma_2^+(1)$ are in $\partial E \cap B_\rho(z)$, we can find a path σ_{12}^+ in ∂E from $\tilde{p}_1$ to $\tilde{p}_2$. We have $t(\tilde{p}_1), t(\tilde{p}_2) \in [k_0 h + \rho, k_0 h + 3\rho]$. We will prove in step 2, that we can find a path $\sigma_{12}^+ \in \partial E$ from $\tilde{p}_1$ to $\tilde{p}_2$ such that for all $p \in \sigma_{12}^+$, $t(p) \in [\rho, k_0 h + 3\rho]$.

Assuming this, we have constructed a path μ^+ in E from p_1 to p_2 which is

$$\mu^+ = \sigma_1^+ \cup \sigma_{12}^+ \cup \sigma_2^+.$$

The arcs $\sigma_1^+(\bar{t}), \sigma_2^+(\bar{t})$ are contained in $T_\rho(\Gamma_+)$. The arcs of σ_1^+ and σ_2^+ from p_1 to F_0 and p_2 to F_0 are disjoint (see proposition 6.1) since $\sigma_1^+ \subset \Sigma_1$ and $\sigma_2^+ \subset \Sigma_2$ in $\mathrm{Tub}^-(F_0)$.

Moreover the paths are quasi-monotone along the segment of Γ_+ in $\mathrm{Tub}(\Gamma_+)$: once the catenoids $\mathrm{C}_\ell(q), q \in \Gamma_+$ have advanced along Γ_+ a distance 2ρ, the paths σ_1^+ and σ_2^+ do not return to the ρ-ball where they started.

If the arcs $\sigma_1^+(\bar{t})$ and $\sigma_2^+(\bar{t})$ remain disjoint for $\bar{t} \leq 1$, we do not change μ^+. If the arcs intersect then at the first point of intersection p_3 we replace μ^+ by the path on σ_1^+ from p_1 to p_3 union the path on σ_2^+ from p_2 to p_3. Such a point p_3 is necessarily outside $F_{-1/2}$.

Step 2: The boundary ∂E. Now we study the boundary of the annulus and the function $t : \partial E \to \mathbb{R}$ the restriction of the third coordinate in the model of the half-plane. We parametrize the boundary curve ∂E by the immersion $C : \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}, C(s) = (x(s), y_0, t(s))$ with period

$$C(s+1) = (x(s+1), y_0, t(s+1)) \longrightarrow (x(s), y_0, t(s) + h).$$

The diameter is defined by

$$G := \sup_{s_1, s_2 \in [0,1]} |t(s_1) - t(s_2)|$$

and choose $k_0 \in \mathbb{N}$ such that $K = k_0 h \geq G$. We consider the intersection of ∂E with a transverse plane to the curve $P(\alpha) := \{(x, y, t) \in \mathbb{R}^3; y = y_0, t = \alpha\}$. Since C is a proper immersed curve, we have a finite number of intersection points

$$C(s) \cap P(\alpha) = \{C(s_1), ..., C(s_\ell)\}.$$

We claim that $(s_i)_{1 \leq i \leq \ell} \in [s_1 - k, s_1 + k]$. To see this we remark that if $s_1 + 1 + k' \geq s \geq s_1 + k' \geq s_1 + k$, we have

$$t(s) - \alpha = t(s) - t(s_1 + k') + t(s_1 + k') - t(s_1) \geq k'\tau - G \geq k_0 \tau - G > 0.$$

Hence independently of the choice of α, two points of ∂E with the same t coordinate are connected by a sub-arc Γ of ∂E with $t(\Gamma) \subset [\alpha - K, \alpha + K]$. Two points of ∂E with coordinate $t_1 \leq t_2$ can be connected in ∂E by a sub-arc Γ with $x(\Gamma) \in [t_1 - K, t_2 + K]$.

Step 3: A loop μ in E. In step 1, we constructed an arc $\mu^+ = \sigma_1^+ \cup \sigma_{12}^+ \cup \sigma_2^+$ which joins the points p_1 and p_2 and $\mu^+ \subset \mathrm{Tub}_\rho(\Gamma_+) \cup \partial E$. Now do this construction in

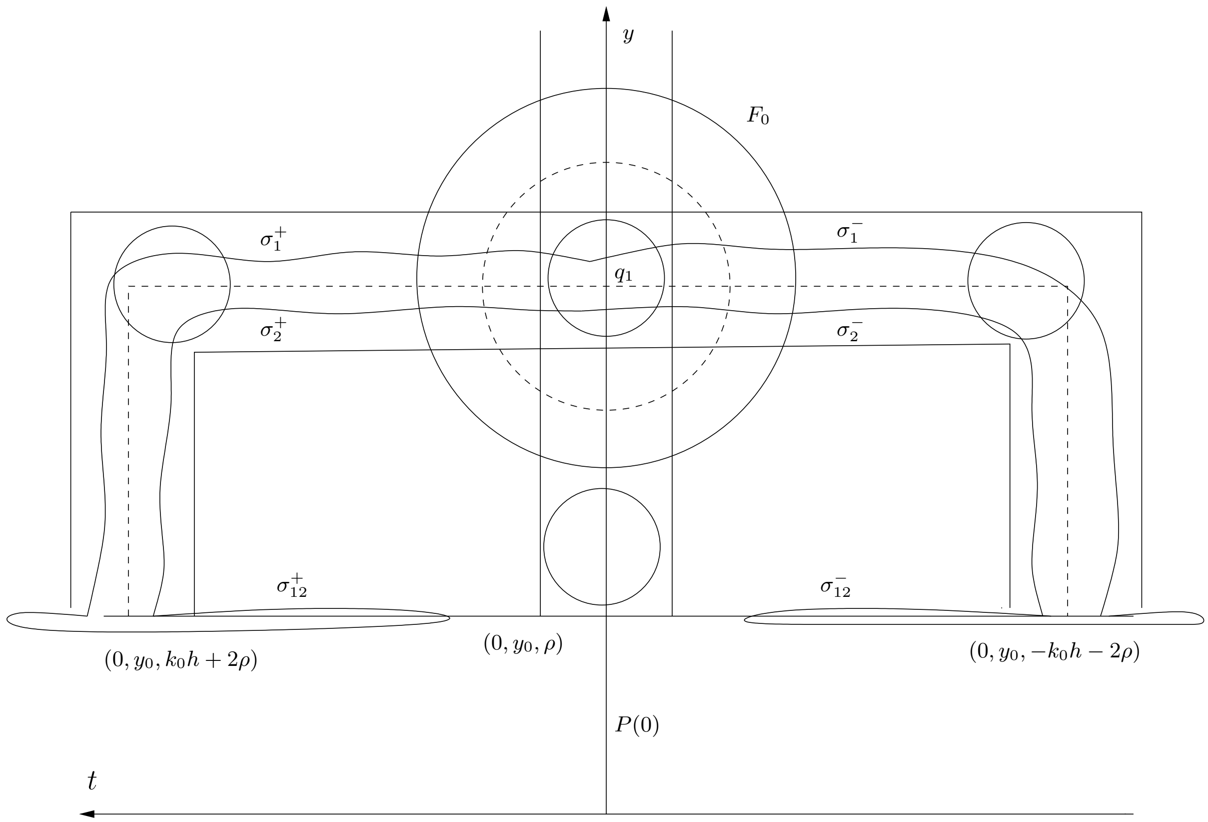

FIGURE 20. Construction of the arc Γ

the half-space $\{x \leq 0\}$ to obtain a path μ^- joining p_1 to p_2 with similar properties. Let Γ_- be the segment from $q_1 = (0, y_1, 0)$ to $(0, y_1, -k_0h - 2\rho)$, together with the segment joining $(0, y_1, -k_0h - 2\rho)$ to $z = (0, y_0, -k_0h - 2\rho)$. Move $B_\rho(q_1)$ in a $\mathcal{C}^1$-monotone manner along Γ_- and we use the Dragging lemma as before to construct arcs $\sigma_1^-(\bar{t}), \sigma_2^-(\bar{t})$ in E. We note by $\text{Tub}_\rho(\Gamma_-)$ the tubular neighborhood of geodesic radius ρ along Γ_-. We follow the arc $\sigma_1^-(\bar{t}), \sigma_2^-(\bar{t})$ up to points of ∂E. As in step 2, we construct the arc σ_{12}^-, so that points $p \in \sigma_{12}^-$ have coordinate $t(p) \subset [-k_0h - \rho, -\rho]$.

Finally we consider; see figure 20

$$\mu^- = \sigma_1^- \cup \sigma_{12}^- \cup \sigma_2^- .$$

and we let μ be the loop $\mu^+ \cup \mu^-$. μ is contained in $\text{Tub}_\rho(\Gamma_+ \cup \Gamma_-)$. If the arcs $\sigma_1^-(\bar{t})$ and $\sigma_2^-(\bar{t})$ remain disjoint for $\bar{t} \leq 1$, we do not change μ^-. If the arcs intersect then at the first point of intersection p_4 we replace μ^- by the path on σ_1^- from p_1 to p_4 union the path on σ_2^- from p_2 to p_4. Such a point p_4 is necessarily outside $F_{-1/2}$.

The end E is an immersed half-plane $X : \Omega = \{(u, v) \in \mathbb{R}^2; v \geq 0\} \to \mathbb{H}^2 \times \mathbb{R}$ with $X(\Omega) = E$. The loop $\mu \subset E$ is immersed and we denote by $\hat{\mu} = X^{-1}(\mu)$ the pre-image of μ in Ω.

In the Dragging lemma, we constructed the arc μ locally and then extended it. The preimage $\hat{\mu}$ is locally embedded in Ω. The arc $\hat{\mu}$ can have self-intersections. If $\hat{p}$ is one of them, we consider the sub-arc γ of $\hat{\mu}$ with end points $\hat{p}$. This sub-arc γ bounds a disk in Ω. We remove these sub arcs to obtain a piecewise $\mathcal{C}^1$ connected curve in Ω without self-intersecting points. This defines a closed Jordan curve which bounds a disk D in Ω. The immersion $X(D)$ is a minimal disk in $\mathbb{H}^2 \times \mathbb{R}$ with boundary an immersed connected curve contained in $\mathrm{Tub}_\rho(\Gamma_+ \cup \Gamma_-) \cup \partial E$. Now we analyze the geometry of the disk $X(D)$.

Consider the plane defined by $P(0) = \{(x, y, t) \in \mathbb{R}^3; y \leq y_0 \text{ and } t = 0\}$. This plane separates $\Gamma_+ \cup \Gamma_-$ in two connected components.

We denote by $\tilde{\mu} = \partial X(D)$ the boundary of the minimal disk. Let $\tilde{\mu}_1 = \tilde{\mu} \cap (\sigma_1^+ \cup \sigma_1^-)$ and $\tilde{\mu}_2 = \tilde{\mu} \cap (\sigma_2^+ \cup \sigma_2^-)$ be the connected components of the loop in E containing p_1 and p_2 respectively. The end points of $\tilde{\mu}_1, \tilde{\mu}_2$ are in different half-spaces determined by $\{x = 0\}$ (one end point has $x > \rho$ and the other $x < -\rho$). Thus the plane $P(0)$ intersects $\tilde{\mu}_1$ and $\tilde{\mu}_2$, each one in an odd number of points.

Now we will obtain a contradiction by proving that $P(0) \cap \tilde{\mu}_1$ is an even number of points. One translates horizontal catenoids $\mathcal{C}_\ell(q)$, $q \in E(0) \cap P(0)$, starting far from μ to see that before $\mathcal{C}_\ell$ touches a ρ-tubular neighborhood of μ, one does not touch the disk $X(D)$. Hence $X(D) \cap P(0)$ is contained in $F_0^- \cap \mathrm{Tub}_\rho(\Gamma_+ \cup \Gamma_-)$.

In F_0^- the sub arc $\tilde{\mu}_1 \subset \Sigma_1$ and $\tilde{\mu}_2 \subset \Sigma_2$ cannot be connected. Hence a connected arc $\gamma \subset X(D) \cap P(0)$ must have end points either in Σ_1 or in Σ_2. This means that there are an even number of point of $\tilde{\mu}_1 \cap P(0)$ on $\partial D = \mu$. This contradicts the odd intersection number of each arc with $P(0)$.

This proves that E is a graph for $y \geq y_0 + 2R$.

Ends of type (p, q), tilted planes. Next we prove the theorem when E is trapped between two tilted (not horizontal) planes $E(p, q)$. We can suppose E is contained a tilted slab S of the form, for some $c_1 > 0$:

$$S = \{(x, y, t) \in \mathbb{R}^3; y > 0 \text{ and } -c_1 \leq p\tau t - qhx \leq c_1\}.$$

Since S is converging to a vertical slab as $y \to \infty$, there is a $y_0 > 1$ so that if $p \in E$, $y(p) \geq y_0$, then the catenoid $C_\ell(p)$ in $B_\rho(p)$, has both its boundary circles outside of S. To see this, we use an isometry which leaves the slab S invariant and takes $p \in E$ to a point $\tilde{p} = (x, y, 0)$. Observe that $S \cap \{|t| \leq 1\}$ is in a vertical slab bounded by $P(-c_2)$ and $P(c_2)$, where c_2 depends on $p\tau$ and qh. Then for any point $\tilde{p} = (x, y, 0)$ with $|x| \leq c_2$, $\mathcal{C}_\ell(\tilde{p}) \subset B_\rho(\tilde{p})$ has boundary circles outside of the slab bounded by $P(d)$ and $P(-d)$ for $d \geq c_2$, and y greater than some y_0 (using that $(x, y, t) \to (\lambda x, \lambda y, t)$ is an isometry). This property is invariant by changing $\tilde{p} = (x, y, 0)$ to $p = (x + p\tau, y, t + qh)$.

We will prove that a sub-end of E is transverse to $\frac{\partial}{\partial x}$ for large y. Suppose this is not the case. We proceed exactly as in the case E is trapped between vertical planes to find $p_1, p_2 \in E \cap B_\rho(q_1)$, q_1 the center of a horizontal catenoid F_0, and p_1, p_2 can not

be joined by a path in E that is inside F_0. The proof is modified in our choice of $\Gamma = \Gamma_+ \cup \Gamma_-$, and a loop in E passing through p_1 and p_2.

We denote by $\vec{u}$ the unit vector director of the straight line $\{(x,y,t) \in \mathbb{R}^3; p\tau t - qhx = 0 \text{ and } y = y_0\}$. For a value $k_0 \in \mathbb{N}$ which depends on the diameter of the periodic boundary curve, we consider Γ_+ be the euclidean segment joining $q_1 = (0, y_1, 0)$ to $q_1 + (k_0h + 2\rho)\vec{u}$, together with the segment joining $q_1 + (k_0h + 2\rho)\vec{u}$ to $z = (0, y_0, 0) + (k_0h + 2\rho)\vec{u}$. We connect the point p_1 and p_2 by an arc in E which stays in a tubular neighborhood of $\Gamma_+ \cup \partial E$.

Let Γ_- be the segment from $q_1 = (0, y_1, 0)$ to $(0, y_1, 0) - (k_0h + 2\rho)\vec{u}$, together with the segment joining $(0, y_1, 0) - (k_0h+2\rho)\vec{u}$ to $z = (0, y_0, 0) - (k_0h+2\rho)\vec{u}$. We connect the point p_1 and p_2 by an arc in E which stays in a tubular neighborhood of $\Gamma_- \cup \partial E$.

Then the argument is the same to obtain a contradiction with $\Gamma = \Gamma_+ \cup \Gamma_-$.

Ends of type $(p, 0)$, horizontal planes. Let E be a half-plane end (lifting of $A \subset \mathcal{M}$) to $\mathbb{H} \times \mathbb{R}$, between the planes $t = \pm d$, with $\partial E \subset \{y = 1\}$, ∂E invariant by the isometry $(x, y, t) \to (x + \tau, y, t)$. By proposition 7.2, we can assume $t \to 0$ on E as $y \to \infty$. So for y_0 large, the sub-end of E given by $y \geq y_0$ is between planes $t = \pm c$ for any small $c > 0$.

Let η be a circle of radius one in $\{t = c\}$ and let η_- be η translated vertically to a circle in $\{t = -c\}$. for c small enough, $\eta \cup \eta_-$ bounds a stable (rotational) annulus F_0. F_0 is a bigraph over $\{t = 0\}$. Now we assume y_0 chosen so that E is between $t = \pm c$ for $y \geq y_0$ and then $\partial F_0 \subset \{t = \pm c\}$.

As in section 6, where E was trapped between two vertical planes and F_0 was a horizontal catenoid, we define $B_\rho(q)$, $\mathcal{C}_\ell$ in the same manner, with $\mathcal{C}_\ell$ a vertical catenoid. We choose y_0 large enough so that E is between $t = \pm\ell/2$ and $\mathcal{C}_\ell$ has its boundary circles in $t = \pm\ell$ for $y \geq y_0 + 3$.

Suppose p is in E , $y(p) \geq y(0) + 3$, and E has a vertical tangent plane at p. Then one places a vertical catenoid F_0 to be tangent to E at p (after a small translation) and one obtains $p_1, p_2 \in E \cap B_\rho(q)$, q the center of F_0, such that p_1, p_2 can not be joined by a path in E that is inside F_0.

For a value $k_0 \in \mathbb{N}$ which depends on the diameter of the periodic boundary curve, we consider Γ_+ be the euclidean segment joining $q_1 = (0, y_1, 0)$ to $(k_0h + 2\rho, y_1, 0)$, together with the segment joining $(k_0h+2\rho, y_1, 0)$ to $z = (k_0h+2\rho, y_0, 0)$. We connect the points p_1 and p_2 by an arc in E which stays in a tubular neighborhood of $\Gamma_+ \cup \partial E$.

Let Γ_- be the segment from $q_1 = (0, y_1, 0)$ to $(-k_0h - 2\rho, y_1, 0)$, together with the segment joining $(-k_0h - 2\rho, y_1, 0)$ to $z = (-k_0h - 2\rho, y_0, 0)$. We connect the point p_1 and p_2 by an arc in E which stays in a tubular neighborhood of $\Gamma_- \cup \partial E$. We apply now the same argument to obtain a contradiction.

Finite total curvature. We proved that a minimal annulus is trapped in Slab and is a killing multigraph outside a compact set $K_0 \subset M \times \mathbb{S}^1$. These graphs are stable, hence they have bounded Gaussian curvature. They are contained in a euclidean slab whose hyperbolic width tends to zero at infinity.

In the horizontal case with A asymptotic to $A_{(p,0)}$, the end A has a limit for its third coordinate. Since the curvature is bounded, A is a vertical graph of a function $f : A_{(p,0)} \to \mathbb{R}$, with f converging to 0 in a $\mathcal{C}^2$ manner. The end A is converging to the cusp $\mathcal{C} \times \{0\}$ and the curve $\mathbb{T}(y) \cap A = \gamma(y)$ is a topological circle converging to a finite covering of a quotient $c(y)/[\psi]$. The curve $\gamma(y)$ has uniform bounded curvature and its length goes to zero. Thus $\int_{\gamma(y)} k_g ds \to 0$ as $y \to \infty$.

In the case of ends of type $(0, q)$ and (p, q), the ends are horizontal multi-graphs on some $A(0, p)$. Since A converges in a $\mathcal{C}^2$ manner to $A(p, 0)$, the curves $\gamma(y) = \mathbb{T}(y) \cap A$ converge to a finite covering of a quotient of a vertical geodesic by the translation $T(h)$. This implies that the curvature of $\gamma(y)$ converges uniformly to zero as $y \to \infty$.

We apply the Gauss-Bonnet formula on an exhaustion of $M \times \mathcal{S}^1$ by a sequence of compact K_n, with boundary of K_n the union of mean curvature one tori $\mathbb{T}_1(n), ..., \mathbb{T}_k(n)$, in each end $\mathcal{M} \subset M \times \mathbb{S}^1$ and $\gamma_{k,n} = \mathbb{T}_k(n) \cap \Sigma$.

$$\int_{K_n \cap S} K dA + \int_{\gamma(k,n)} k_g ds = 2\pi\chi(\Sigma).$$

When $n \to \infty$, the integral of the curvature on $\gamma(k, n)$ tends to zero and we obtain the finite total curvature formula

$$\int_S K dA = 2\pi\chi(\Sigma).$$

9. Proof of the theorem in N

Now we complete the proof of the Theorem 1.1 when the ambient space is N. The idea is the same as in $M \times \mathbb{S}^1$. Let A be an annular end in $\mathcal{M}(-1)$, minimal and properly immersed. By Lemma 4.1 (the same proof) we can suppose $A \subset \cup_{y \geq 1} \mathbb{T}(y)$, ∂A is an immersed closed curve and A is transverse to $\mathbb{T}(1)$ along ∂A. Let E be a connected lift of A to $\mathbb{H}^3$, so $\partial E \subset \{(x, 1, t) \in \mathbb{R}^3\}$. Observe that E is a half-plane, not an annulus. Suppose, on the contrary that E is an immersed annulus, so $\partial E \subset \{(x, 1, t) \in \mathbb{R}^3\}$ is compact. Let D be the convex hull of $\{(x, 0, t) \in \mathbb{R}^3; (x, 1, t) \in \partial E\}$ in the $y = 0$ plane. Let L be a line of the plane $Q = \{y = 0\}$, disjoint from D.

Let C be a small circle in Q in the half-space $\mathcal{H}$ of $Q - L$ disjoint from D. C bounds a totally geodesic hyperbolic plane in $\mathbb{H}^3$ (it is a hemisphere orthogonal to Q along C in our model). For C small, this plane is disjoint from E. Let the circle C grow in $\mathcal{H}$ and converge to L. By the maximum principle, there is no first contact point of the planes bounded by these circles with E (the planes do not touch ∂E). Since the hyperbolic planes bounded by the circles converge to $L \times \mathbb{R}^+$, it follows that E is on one side of $L \times \mathbb{R}^+$. Hence E is contained in the cylinder $\mathrm{Cyl} = \{(x, y, t) \in \mathbb{R}^3; (x, 0, t) \in \partial D, y > 0\}$.

For y large, the diameter of Cyl tends to zero; i.e. the diameter of $\mathrm{Cyl} \cap \{y = \mathrm{const}\}$ tends to zero. So we could touch E by a catenoid at an interior point of E; a contradiction.

Now we know E is a half-plane. After an isometry of $\mathbb{H}^3$, we can assume ∂E is invariant under the parabolic isometry: $(x, y, t) \to (x + \tau, y, t)$, and $\partial E \subset \{y = 1\}$. so the t coordinate is bounded on ∂E. The same convex hull argument as in the previous annular case, then shows the t coordinate has the same bound on E; $|t| \leq c$, for some $c > 0$, (one takes L to be a horizontal line in $\{y = 0\}$, above height c, and considers circles C in $\{y = 0\}$ above L. When C converges to L in $\{y = 0\}$, the hyperbolic planes in $\mathbb{H}^3$ bounded by C, are disjoint from E and converge to $L \times \mathbb{R}^+$). So E is trapped between two horizontal planes $t = \pm c$.

The distance between these horizontal planes tends to zero as $y \to \infty$. Now we will prove that for y large, the killing field $\frac{\partial}{\partial t}$ is transverse to E. hence a sub-end of E has bounded curvature. This will complete the proof as follows. The sub-end is a vertical graph over the plane $t = 0$, that converges to zero in the $\mathcal{C}^2$-topology. The graph function is the distance to the plane $t = 0$. Thus the geodesic curvature of the curve in E, given by $\mathrm{Cyl} = E \cap \{y = \text{constant}\}$ is bounded. Also the length of this curve C_y tends to zero in C_y modulo $(x, y, t) \to (x + \tau, y, t)$. This yields the formula for the finite total curvature of Σ in N: apply Gauss-Bonnet to the compact part of Σ bounded by the curves C_y in the ends and let $y \to \infty$.

Thus it suffices to prove E is transverse to $\frac{\partial}{\partial t}$ for y large. The proof of this is the same as in section 8, for an end trapped between two horizontall planes. More precisely, for an end E in $\mathcal{M}$ between two horizontal planes that are close, the distance between the planes $|t| = c$ tends to zero as $y \to \infty$, so one can put a vertical catenoid F_0, whose boundary circles are of radius one and in the horizontal planes $|t| = d > c$, when the center q of F_0 has $y(q)$ larger than some y_0.

One chooses ρ, ℓ as in Sections 6 and 8, and using the Dragging lemma, one shows that if E has a vertical tangent plane at p, $y(p)$ large, then one finds $p_1, p_2 \in B_\rho(q) \cap E$, that can not be joined by a path in $E \cap F_0^-$. One defines $\Gamma = \Gamma_+ \cup \Gamma_-$ and the same proof now gives a contradiction.

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